MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

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SIR DAVID GILL, K.C.B., F.R.S., PRESIDENT, in the Chair.

The President said:—

We have met to-day under circumstances of the deepest gloom, and I feel sure that you will all agree with your Council's resolution that there shall be no other business done to-day than that which love and duty to our late Sovereign require, and which the hand of death has laid upon us.

It is needless for me to say how much we and the world at large will miss the wise and tactful ruler who has passed away.

The Addresses which have been prepared in Council so completely voice what I am sure everyone present feels, that I will now read them and ask you to concur.

To the King's Most Excellent Majesty.

May it please Your Majesty:

We, the President, Council, and Fellows of the Royal Astronomical Society, beg leave humbly to approach Your Majesty with the expression of our sorrow and respectful sympathy on the death of our beloved Sovereign, King Edward the Seventh.

We have reverenced the kingly wisdom with which He guided the affairs of the nation, and exerted His influence in maintaining the peace of the world. His constant care and self-sacrifice for the welfare of His subjects have commanded our ever-increasing devotion, and we have rejoiced that His genial influence has been felt among all nations.

It was our privilege that He was graciously pleased to be the Patron of our Society, an honour which was also conferred upon us by our beloved Queen Victoria, and His Majesty King William the Fourth.

We further beg leave to offer our dutiful congratulations on the accession of Your Majesty to the throne of Your Ancestors, being fully assured of the continuance of that active devotion to the welfare and unity of the Empire which You have shown as Prince of Wales.

We pray that Your Majesty with Your Royal Consort may, under the blessing of God, be granted both health and happiness and a long and prosperous reign.

To her Majesty the Queen Mother.

May it please Your Majesty:

We, the President, Council, and Fellows of the Royal Astronomical Society, beg leave humbly to approach Your Majesty with the expression of our sorrow at the death of our beloved Sovereign, King Edward the Seventh, the Patron of our Society.

We beg to tender our dutiful sympathy to Your Majesty, and to assure You that it is our heartfelt prayer that Almighty God will vouchsafe You consolation in Your bereavement.

These Addresses were agreed to, the Fellows present upstanding in silence during the reading of them by the President.

The President then said:—

We have another sorrowful duty to perform, viz., to record our sense of the irreparable loss which we, in common with the whole scientific world, have sustained by the death of Sir William Huggins.

To a very large number of us he was a dear personal friend; to all of us his life has been a noble example in its purity and earnest singleness of purpose in the quest of truth. By his genius and industry during a long life he has bequeathed a priceless boon to the Science of Astronomy, and to Science in general.

The heart of every Fellow present, I feel sure, goes out in sympathy to Lady Huggins in this sore and sudden bereavement, and I venture to think that you will all give your silent assent to the Resolution of the Council which I will now read.

The Fellows present thereupon rose and stood in silence during the reading of the Resolution by the President.

The Council have learned with the deepest regret of the death of Sir William Huggins, and desire to record their sense of the great loss which the Society itself, and Science in general, have thus sustained.

As Fellow of the Society since 1854, as a Member of the Council since 1864, as Secretary in 1867–72, as Foreign Secretary in 1873–75, as President in 1876–78, and as Foreign Secretary from 1883 to the present time, he has rendered services of the greatest value to the Society. His wide knowledge and sound judgment were ever at its disposal.

But it is on the higher ground of his having been the pioneer in all those branches of research, now termed astrophysics, that he has the greatest claim to respect and admiration.

The Council, in requesting the President to convey their sympathy to Lady Huggins on her bereavement, desire him to say how much they have reason to be proud and thankful for the noble life's work of her husband, with whom she has actively collaborated for so many years.

The Meeting then adjourned.

The following are the papers which would in the ordinary course have been read at the meeting on May 13:—

On the smallest visible phase of the Moon.

By J. K. Fotheringham, M.A., D.Litt.

It is well known that many calendars, both in ancient and modern times, have reckoned the month from the first appearance of the Moon, and frequent complaints have been made by students of chronology of the absence of any definite information upon the astronomical conditions which govern the visibility of the Moon to the naked eye.* In an article which appeared in the *Journal of Philology* in 1903 † I suggested that "in order to calculate the true date of the phasis we ought to have a table of the requisite depression of the Sun below the horizon at moonset, or of the requisite altitude of the Moon above the horizon at sunset for different angular distances of the Moon from the Sun." But I was not then aware that the observations on which such a table might be constructed were already provided in August Mommsen's *Chronologie* (1883), pp. 69–80. Mommsen here records forty-eight positive observations of first sight of new Moon made by Julius Schmidt at Athens, one such observation (No. 13 below) made by Julius Schmidt at Corinth, and one such (No. 75) made by Mommsen himself at Athens, eighteen negative observations of new Moon (*i.e.* occasions when the Moon might have been expected to be visible, but was not) made by Julius Schmidt at Athens, one such (No. 19) made by Julius Schmidt at Troy, and one such (No. 74) made by Mommsen at Athens, two positive observations of last sight of old Moon (Nos. 36, 47) made by Julius Schmidt at Athens, one such (No. 43) made by Friedrich Schmidt at Athens, two such (Nos. 72, 76) made by Mommsen at Athens, and one negative observation of old Moon (No. 73) made by Mommsen at Athens. The total number of observations is 76. It is probable that all Julius Schmidt's negative observations were made on fine evenings, as they profess to have been made on evenings when the Moon might have been expected to be visible. Mommsen's negative morning observation (No. 73) might, according to his own suggestion, be due to an obscuration of the Moon by Hymettus, but Mommsen himself rejects this suggestion. The observation was made on a walk which extended till the disappearance of the stars. This raises a doubt whether, if the walk had been prolonged a few minutes longer, it might not have had a different effect. Observation No. 74 was made on a cloudy evening, but Mommsen notes that there was a gap in the clouds through which the Moon, if visible, might have been seen. This

* I may instance my paper, "The Date of the Crucifixion," in *The Journal of Philology*, xxix. (1903), pp. 104, 105; Ginzel, *Handbuch der mathematischen und technischen Chronologie*, i. (1906), p. 93; Weissbach, *Zum babylonischen Kalender*, Hilprecht Anniversary Volume (1909), p. 286.

† *Ubi supra.*

measure of uncertainty attaching to Nos. 73, 74 is the more tantalising, because these observations would otherwise have been among the most valuable of the whole series. For fuller information about observations 1–33, see an article by Schmidt in *Astronomische Nachrichten*, Band 71 (1868), pp. 201 ff.; and for fuller information concerning the whole series, see Mommsen's *Chronologie, ubi supra.*

The whole series of observations was sent by Mommsen to Bruhns, several of whose comments are published by Mommsen. Mommsen expresses himself disappointed that the study of these observations should not have led to the discovery of an exact method of determining the last visible phase of the old Moon and first visible phase of the new. Bruhns's failure to discover such a method was certainly not due to any defect in the observations, and I can only suppose that he did not perfectly understand the purpose of Mommsen's inquiry.

In the following table I give the civil date of each observation, the true altitude of the Moon at sunset for evening observations and at sunrise for morning observations, the true difference in azimuth of Sun and Moon for the same moment, and the result of the observation. The date and result are taken from Mommsen, while I have computed the altitude and azimuths. I have thought it unnecessary to complicate the calculation by introducing the lunar parallax, because the parallax in altitude is practically constant for any given altitude, while the parallax in azimuth is small, and small changes in azimuth produce no perceptible difference in the result.

Reference No.	Date.	Altitude.	Difference in Azimuth.	Result.
		°	°	
1	1859 July 1	12·7	10·0	Visible.
2	Oct. 27	6·1	20·5	Visible.
3	1860 Jan. 23	6·0	3·2	Not visible.
4	Feb. 23	20·2	2·4	Visible.
5	June 20	15·4	12·5	Visible.
6	1861 March 12	12·8	1·8	Visible.
7	Aug. 7	5·2	15·2	Not visible.
8	Aug. 8	11·2	27·1	Visible.
9	Sept. 7	12·9	36·4	Visible.
10	Oct. 5	5·3	19·4	Not visible.
11	Nov. 4	12·7	25·2	Visible.
12	Dec. 3	13·6	16·6	Visible.
13	1862 Jan. 1	12·3	7·1	Visible.
14	March 31	15·9	0·4	Visible.
15	Apr. 29	8·6	0·1	Not visible.
16	July 28	8·6	20·5	Not visible.
17	1864 Jan. 10	18·4	6·5	Visible.
18	March 9	21·1	2·3	Visible.

Reference No.	Date.	Altitude.	Difference in Azimuth.	Result.
19	1864 May 6	7°·9	4°·6	Not visible.
20	June 6	18·5	20·0	Visible.
21	Aug. 4	8·7	22·1	Not visible.
22	Sept. 3	11·4	25·8	Visible.
23	Nov. 1	13·7	18·7	Visible.
24	1865 Jan. 28	17·8	3·4	Visible.
25	March 28	20·0	5·4	Visible.
26	Apr. 26	13·6	7·2	Visible.
27	June 24	9·2	16·0	Not visible.
28	July 24	9·7	21·3	Visible.
29	Oct. 21	12·8	17·3	Visible.
30	1866 Jan. 17	10·7	0·8	Not visible.
31	Apr. 16	18·0	9·2	Visible.
32	1867 Feb. 5	10·5	1·8	Not visible.
33	Nov. 27	13·7	9·6	Visible.
34	1868 June 22	19·7	22·8	Visible.
35	1869 May 12	9·1	9·4	Not visible.
36	1870 July 25 M.	32·8	23·3	Visible.
37	1871 Feb. 20	11·4	8·9	Visible.
38	Apr. 20	8·4	7·0	Not visible.
39	May 20	11·2	8·0	Not visible.
40	June 18	5·8	4·6	Not visible.
41	June 19	14·2	10·9	Visible.
42	Aug. 17	12·9	12·1	Visible.
43	Sept. 14 M.	8·8	2·8	Visible.
44	1872 June 7	15·3	9·0	Visible.
45	July 6	9·8	5·6	Not visible.
46	Sept. 4	13·1	14·3	Visible.
47	Sept. 30 M.	28·7	2·1	Visible.
48	Oct. 3	8·9	9·0	Not visible.
49	Oct. 4	13·6	20·3	Visible.
50	Dec. 31	12·2	15·3	Visible.
51	1873 Apr. 27	9·1	4·2	Not visible.
52	May 27	15·2	6·5	Visible.
53	Dec. 20	4·9	10·4	Not visible.
54	1874 Apr. 17	15·3	4·2	Visible.
55	1875 July 4	17·4	12·2	Visible.
56	1876 Feb. 26	16·1	5·5	Visible.
57	June 22	11·9	4·6	Not visible.
58	July 22	14·6	15·9	Visible.
59	1877 March 16	18·0	5·3	Visible.

Reference No.	Date.	Altitude.	Difference in Azimuth.	Result.
		°	°	
60	1877 June 12	14·7	6·4	Visible.
61	Nov. 7	11·1	27·2	Visible.
62	Dec. 6	10·5	18·1	Visible.
63	1878 Jan. 5	17·8	14·9	Visible.
64	June 2	18·6	8·7	Visible.
65	July 1	12·2	10·4	Visible.
66	July 31	12·2	21·7	Visible.
67	Oct. 27	7·9	23·0	Visible.
68	Nov. 26	15·8	26·2	Visible.
69	1879 May 23	24·9	12·3	Visible.
70	June 22	23·5	24·9	Visible.
71	July 22	20·0	37·1	Visible.
72	Dec. 11 M.	21·6	25·9	Visible.
73	Dec. 12 M.	11·2	13·3	Not visible.
74	Dec. 14	10·8	12·1	Not visible.
75	Dec. 15	21·1	21·6	Visible.
76	1880 Jan. 10 M.	13·4	19·4	Visible.

These observations give a very clear dividing line between the conditions of positive and negative observations.

The following table satisfies all the observations except Nos. 2 and 43.

True Difference in Azimuth at Sunset (or Sunrise).	Minimum True Altitude at Sunset (or Sunrise) for Moon to be visible same evening (or morning).
°	°
0	12·0
5	11·9
10	11·4
15	11·0
20	10·0
23	7·7

The two discordant observations are both positive, where a negative observation might have been expected. One (No. 43) is a morning observation, and may be evidence that the same rule does not apply to morning and evening observations. As Mommsen only records six morning observations, the evidence for morning conditions is much weaker than for evening conditions. That there should be only one discordant evening observation among so many is remarkable, and seems to show that, given a clear sky, *the problem is almost purely astronomical, and not atmospheric.* As the solution given above takes into account nothing except the relative positions of the Sun, Moon, and horizon, it is independent of differences in latitude, and *ought therefore to be applicable to any place,* subject to a slight modification for permanent differences in the clearness of the air. The Moon should,

unless obscured by clouds, appear on the first evening when her altitude at sunset has reached the height required by the difference in azimuth.

The rapidity with which the Minimum Altitude diminishes when the difference in azimuth exceeds 20° is worthy of attention. The observations under discussion include five (Nos. 2, 16, 21, 28, 67) where a difference of more than 20° in azimuth is combined with an altitude of less than 10°. In three of these five instances the Moon was visible. On the other hand, No. 43 is the only instance of the Moon being visible where the difference in azimuth is less than 20° and the altitude is less than 10°. A rough approximation to the conditions of visibility of the Moon found above is given by the formula

$$\text{Minimum Altitude} = 12^\circ{\cdot}0 - 0^\circ{\cdot}008\ Z^2,$$

where Z = the number of degrees in difference of azimuth; but the summary table given above is more accurate.

The only previous attempt to give an exact rule for determining the smallest visible phase of the Moon, so far as I know, is that of the Jewish philosopher Maimonides in his treatise on the Sanctification of the New Moon, best studied in von Littrow's article in *Sitzungsberichte der Wiener Akademie, Math.-Naturw. Classe*, lxvi. (1872), pp. 459–480. Maimonides makes the result depend on two variables, the true elongation of the Moon and the apparent "angle of vision," both computed for a time which is on the average twenty minutes after sunset. If "angle of vision" means difference of Sun and Moon in zenith-distance, his rule has very nearly the same effect as mine, but gives slightly lower minimum altitudes, from which it may follow that observations are slightly easier at Jerusalem than at Athens. I do not, however, sufficiently understand his arithmetical method to be able to state with certainty whether this is what he intends by "angle of vision."

It is to be hoped that this solution will prove valuable to chronologists and to astronomers who may be consulted by chronologists. Dates dependent on the first appearance of the Moon are found in ancient Egypt, Palestine, and Babylonia, and in the Mohammedan world down to the present day. The earliest Greek calendars also began the month with the appearance of the Moon, but we have few, if any, definite dates belonging to these calendars. All these countries have a very clear atmosphere, though they may differ slightly one from another. Hitherto it has been the practice to assume that the Moon becomes visible on the first evening when she is more than 30 hours old at sunset. Some important dates in Egyptian and early Christian history have been made to depend on this, and I shall be glad if this paper makes it possible to fix them on a surer foundation.

I have to thank Professor Turner for some labour-saving suggestions in the preparation of this paper.

24 *The Avenue, Muswell Hill,*
London N.: 1910 *May* 9.

On the Mass-Determination of Parallaxes.* By Karl Pearson, F.R.S., and Julia Bell, M.A. (Plates 16, 17.)

(1) In the course of some recent work we were very desirous of obtaining the value of the parallax constant, V_D, the coefficient of variation of stellar distance. The constant can be found indirectly in a number of ways, but its direct determination depends on the knowledge of a very large number of parallaxes. In endeavouring to determine it directly we have examined the memoir of Professor Kapteyn and Dr. de Sitter, with its details of the parallaxes of 3650 stars.† We found for this material that the values of V_D and V_π ‡ came out very differently from those obtained from other series of parallaxes, from values found indirectly, and from the theoretical values which result from various hypotheses as to the distribution of the stars. They are in fact of a wholly different order. This led us to table various points as to the frequency distributions of the Kapteyn parallaxes, which it is possible may be of some general interest.

It will be remembered that Kapteyn tables two parallaxes,—Π_0, the final reduction value without magnitude corrections, and Π, the final value including corrections for magnitude errors. We have dealt with both these quantities.

(2) Before discussing the frequency distributions, we may note that the same stars occur in some 406 cases on two plates. Table I. is a correlation table for the parallax of the same star as determined from different plates. Taking A and B to represent the two measurements we have, all measurements being in seconds of arc, §

$$m_A = -\cdot00207, \qquad \sigma_A = \cdot05595,$$
$$m_B = -\cdot00071, \qquad \sigma_B = \cdot05105,$$
$$r_{AB} = \cdot301 \pm \cdot0304.$$

The table, and these constants for it, show that the determination of a parallax from one plate gives very little idea of what its value will be found from a second plate. We have on the average—

$$\frac{\text{variability of second measure of parallax}}{\text{variability of all parallaxes}} = \sqrt{1 - r_{AB}^2} = \cdot95.$$

To determine how far this was due to the measurer or the plate, we correlated the cases, 28 in number, in which duplicate parallaxes

* The word "mass" is used here in the technical statistical sense of the *Theorie der Massenerscheinungen* of Professor W. Lexis. Kapteyn himself uses "wholesale."

† "The Parallaxes of 3650 Stars of Different Galactic Latitudes, derived from Photographic Plates prepared by Professor Anders Donner," *Publications of the Astronomical Laboratory at Groningen,* No. 20, Groningen, 1908.

‡ See *Royal Soc. Proc.*, June 1910, for the application and significance of these constants of parallax.

§ All duplicate measurements on the same plate excluded.

had been measured on the same plate. Considering the paucity of numbers, the result was very good: we found—

$$r_{AA'} = \cdot 845 \pm \cdot 037,$$

which denotes a reasonably close value from duplicate measurements on the same plate. We now proceeded to determine whether r_{AB} for measurements on different plates would be substantially modified if we took Kapteyn's Π values instead of Π_0. There resulted the distribution given in Table II. for 399 stars. The constants obtained were—

$$m_A = +\cdot 00021 \qquad \sigma_A = \cdot 04820$$
$$m_B = +\cdot 00046 \qquad \sigma_B = \cdot 05254$$
$$r_{AB} = \cdot 318 \pm \cdot 030.$$

Thus within the limits of the probable error the correlation is identical, and duplicate values of Π from different plates are as slightly related as duplicate values of Π_0.

(3) We now turn to the frequency distributions of Π_0 and Π.* These are given in Table III. From these frequencies we determined the following constants of the frequency distributions:—

$$m_{\pi 0} = -\cdot 0002 \pm \cdot 0006 \qquad m_\pi = +\cdot 0036 \pm \cdot 00053$$
$$\sigma_{\pi 0} = \cdot 04952 \pm \cdot 00039 \qquad \sigma_\pi = +\cdot 04695 \pm \cdot 00037$$

These are measured in seconds of angle. The following are measured in a unit $= \cdot 02''$:—

$$\mu_2 = 6\cdot 1306 \qquad \mu_2 = 5\cdot 5097$$
$$\mu_3 = -\cdot 1906 \qquad \mu_3 = -\cdot 6188$$
$$\mu_4 = 130\cdot 5996 \qquad \mu_4 = 100\cdot 9693$$
$$\beta_1 = \cdot 00016 \pm \cdot 00005 \qquad \beta_1 = \cdot 0023 \pm \cdot 0006$$
$$\beta_2 = 3\cdot 4748 \pm \cdot 1180 \qquad \beta_2 = 3\cdot 3261 \pm \cdot 0946$$

μ_2, μ_3, μ_4 are the usual first three moment-coefficients, and β_1 and β_2 the fundamental constants μ_3^2/μ_2^3 and μ_4/μ_2^2. Now for a Gaussian normal curve of errors, $\beta_1 = 0$ and $\beta_2 = 3$. Also if we suppose a Gaussian distribution, we should have for probable errors †—

$$\text{p.e. of } \sqrt{\beta_1} = \cdot 0276 \text{ for } \Pi_0 \text{ and for } \Pi,$$
$$\text{and } \sqrt{\beta_1} = \cdot 0127 \text{ for } \Pi_0 \text{ and } \cdot 0480 \text{ for } \Pi.$$

Thus in both cases the value of $\sqrt{\beta_1}$ is compatible with a Gaussian distribution of errors. But for the Gaussian †—

$$\text{p.e. of } \beta_2 = \cdot 0552 \text{ for } \Pi_0 \text{ and for } \Pi,$$

and the deviations of β_2 from 3 are nearly 9 and 6 times respectively this probable error. It appears, therefore, that neither distribution is truly Gaussian; we have to deal with *symmetrical* dis-

* Suspected plates and duplicate measurements omitted.

† Pearson, *Phil. Trans.*, vol. 198, A, p. 278.

tributions which are non-Gaussian. Examining Rhind's diagram * to determine the type of frequency, we see that in both cases it falls practically into Type VII., *i.e.* is of the type—

$$y = y_0\left(1 + \frac{x^2}{a^2}\right)^{-n}$$

or in the usual form

$$y = \frac{N}{\sqrt{2\Pi}\,\sigma}\,\frac{\Gamma(m_2+1)}{\sqrt{m_2-\tfrac{1}{2}}\,\Gamma(m_2+\tfrac{1}{2})}\left(1+\frac{x^2}{2\sigma^2(m_2-\tfrac{1}{2})}\right)^{-(m_2+1)}$$

where

$$m_2 = \frac{3(\beta_2-1)}{2(\beta_2-3)}.$$

Numerically with the unit for $x = \cdot 02''$ we find—

For Π_0: $\quad y = 606{\cdot}82\left(1+\dfrac{x^2}{89{\cdot}7324}\right)^{-8{\cdot}8184}$. (Plate 16.)

For Π: $\quad y = 631{\cdot}70\left(1+\dfrac{x^2}{112{\cdot}393}\right)^{-11{\cdot}6996}$. (Plate 17.)

These are clearly, remembering that for p large $\left(1+\dfrac{x^2}{p}\right)^{-p} = e^{-x^2}$, not far from the Gaussian curve, but they differ just sensibly from it.

These curves were plotted and then integrated mechanically. The values of the frequencies obtained for each sub-range are given in the last two rows of Table III. under the headings "Calculated." It will be seen by examining these frequencies and also the diagrams that the fit is very close. Applying the customary test for goodness of fit,† we find that for Π_0, P = ·975, and for Π, P = ·736. In other words, for random sampling of material following the above theoretical distributions, we should get a worse fit than the observed in the first case, in 975 out of 1000 trials, and in the second case, in 736 out of 1000 trials. It is clear, therefore, that the above curves express excellently the observed quantities.

Further, we see that for the case of Π_0 the mean parallax is essentially zero, and we cannot assert that it certainly differs from zero in the case of Π. Thus Kapteyn's parallaxes practically give distributions about zero parallax with errors in excess and defect equally probable. If his high parallaxes +·13 to +·16 be considered significant, it must be admitted that they occur with exactly the same frequency as his negative parallaxes −·13 to −·16. The value of $V_{\pi_0} = \sigma_{\pi_0}/m_{\pi_0}$ is impossible; the value of $V_\pi = 13$ is far higher than that found for any other series of parallaxes, and quite incompatible with any even approximately uniform distribution of the stars through space.‡

* In *Biometrika*, vol. vii. p. 131 and vol. iv. p. 174.

† *Biometrika*, vol. i. p. 155.

‡ See Pearson, "On the Improbability of a Random Distribution of the Stars in Space," *Royal Soc. Proc.*, June 1910.

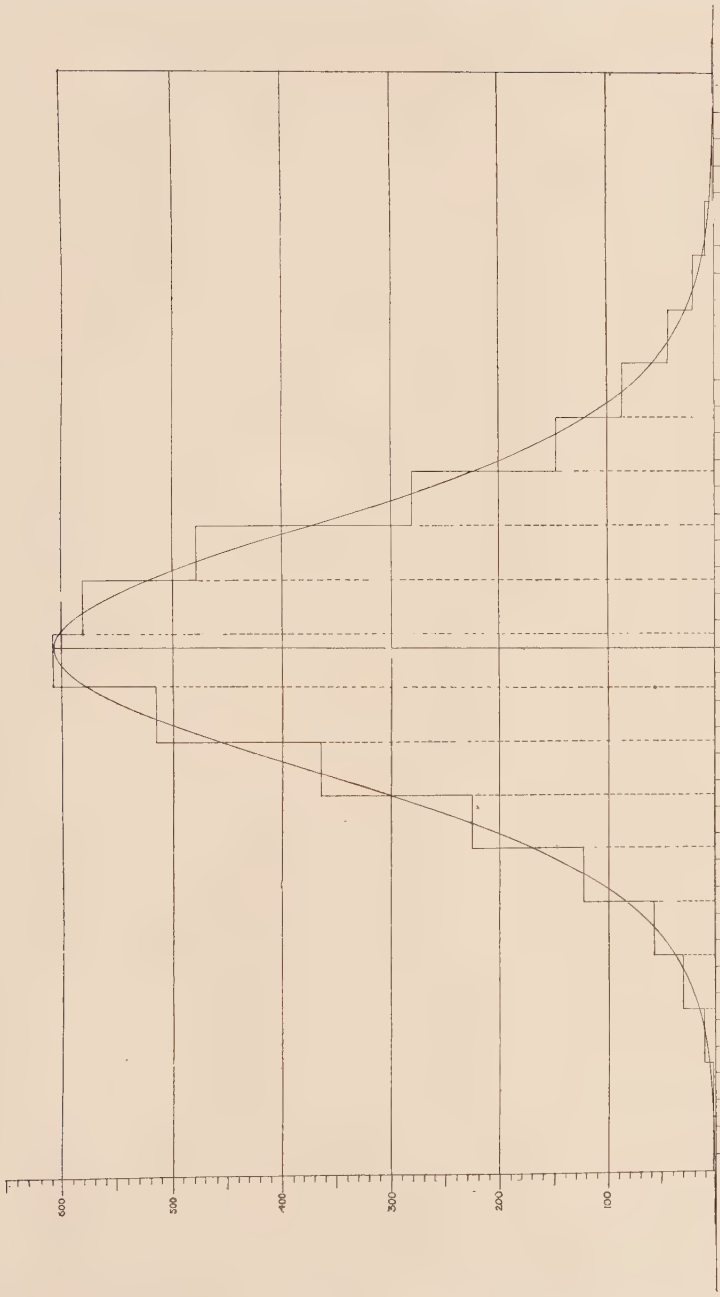

DIAGRAM I.—DISTRIBUTION OF PARALLAX π_0.

Scales:—Horizontal, 1 division = ·01″.
Vertical, 1 division = 10 parallaxes.

DIAGRAM II.—DISTRIBUTION OF PARALLAX π.

Scales :—Horizontal, 1 division = ·01″.
Vertical, 1 division = 10 parallaxes.

The points that seem necessary to emphasise are the sensibly *symmetrical* distribution round the zero value of Π_0, a result which would flow from accidental errors, but hardly from parallax. If we take the 406 cases occurring on the two different plates, it is true that there is a small correlation between the values found, but it is so small that it would be quite impossible to predict the value found from one plate from that found from the other, and the mean value of the parallaxes from each of these sets of observations is again zero. Further, if we take σ_{A-B}, the standard deviation of the difference of two determinations, which is equal to

$$(\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B r_{AB})^{\frac{1}{2}},$$

we find it equal to

$$\cdot 0634'',$$

while $\sqrt{\sigma_A^2 + \sigma_B^2} = \cdot 0758$,

leading to probable errors of

$$\cdot 0428 \text{ and } \cdot 0511 \text{ respectively}$$

with a difference of $\cdot 008''$.

Kapteyn gives a difference (p. 27) of twice this value ($\cdot 017''$) from a somewhat different series. But an examination of the correlation table shows us that the fact that a star has a *negative* parallax on one plate renders it probable that it will have a *negative* parallax on the other; and this, although the relationship is exceedingly small, must be borne in mind when estimating, whether we are dealing with a true parallax or correlated accidental errors peculiar to two photographs of the same star. The general difficulty about accepting a symmetrical distribution around zero as a parallax distribution appears to militate against the view that even the small relationship found is really a true parallax relation. If it indeed is, it is not sufficient to allow of any determination from it of mean parallax and mean variability of parallax (V_π or V_D), such as we had in view in reducing these measurements.

TABLE I.

Parallaxes of Same Star derived from Different Plates. Π_0.

A. MEASUREMENT.

B. MEASUREMENT.

	−·17 −·16	−·15 −·14	−·13 −·12	−·11 −·10	−·09 −·08	−·07 −·06	−·05 −·04	−·03 −·02	−·01 ·00	+·01 +·02	+·03 +·04	+·05 +·06	+·07 +·08	+·09 +·10	+·11 +·12	+·13 +·14	+·15 +·16	
−·15 −·14	...	...	...	...	...	...	...	...	1	...	...	...	...	...	...	...	...	1
−·13 −·12	...	1	...	...	...	1	...	...	...	...	...	...	...	...	...	...	...	2
−·11 −·10	...	...	...	1	2	...	1	2	3	1	...	1	...	...	...	...	...	11
−·09 −·08	...	1	...	...	1·5	1	4·5	4·5	1·5	1	2	2	3	...	...	1	·5	23·5
−·07 −·06	1	1	·5	...	2	3	4·5	4·5	...	2	1	...	...	...	...	...	·5	20
−·05 −·04	...	...	1·5	1	4	...	9·5	4	8	5	2·5	1	1	1	...	...	...	38·5
−·03 −·02	...	...	1	3	3·5	9·5	5	6	8	11·75	2·25	4	3	...	...	...	...	57
−·01 ·00	...	...	..	1	2	2·5	4	12	10·5	14·25	8·25	4·5	3	1	1	...	...	64
+·01 ·02	...	1	...	2	4	3·5	8	4·5	8·5	15·75	8·25	4	2·5	4	...	1	...	67
·03 ·04	...	...	...	2	...	1	3·5	8	6·5	7·25	12·25	4·5	5	1	1	...	...	52
·05 ·06	...	...	1	...	1	1	2	4	7	4	5·5	3	3·5	...	·5	...	·5	33
·07 ·08	...	1	...	1	...	1	1	1	1	5·5	4·5	1	...	3	·5	2	·5	23
·09 ·10	...	...	...	...	...	...	...	...	...	...	3	1	...	...	1	...	...	5
·11 ·12	...	...	...	...	...	...	...	...	...	1	1	2	1	...	...	...	...	5
·13 ·14	...	..	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
·15 ·16	...	...	...	...	1	...	...	...	...	1	...	...	...	...	2	...	...	4
	1	5	4	11	21	23·5	43	50·5	55	69·5	50·5	28	22	10	6	4	2	406

TABLE II.

Parallaxes of Same Star derived from Different Plates. II.

A. MEASUREMENT.

	−·15 −·14	−·13 −·12	−·11 −·10	−·09 −·08	−·07 −·06	−·05 −·04	−·03 −·02	−·01 ·00	+·01 +·02	+·03 +·04	+·05 +·06	+·07 +·08	+·09 +·10	+·11 +·12	+·13 +·14	+·15 +·16	
−·17 − ·16	...	...	...	...	1	...	...	...	...	...	...	...	...	...	..	...	1
−·15 − ·14	...	...	...	...	...	...	...	...	1	1	...	...	...	...	...	...	2
−·13 − ·12	...	...	1	...	1	...	...	...	...	...	...	...	...	...	...	...	2
−·11 − ·10	...	2	...	1	1	2	1	...	1	1	2	...	...	...	...	...	11
−·09 − ·08	...	...	...	1	1	6	2	3	...	1	1	2	...	...	...	...	17
−·07 − ·06	1	...	1	1	3	5	5	1	4	4	1	...	...	...	...	...	26
−·05 − ·04	...	...	2	2	2	5	6	2	11	6	1	1	...	...	...	...	38
−·03 − ·02	...	1	1	1	2	11	10	8	13	4	3	1	...	..	...	...	55
−·01 ·00	...	...	1	4	3	4	8	7	14	10	2	3	1	...	...	...	57
+·01 ·02	...	1	2	1	4	7	9	7	8	8	6	2	2	...	...	...	57
·03 ·04	...	...	...	...	3	2	1	8	14	8	6	6	1	1	...	1	51
·05 ·06	...	1	...	1	1	2	5	6	7	11	5	1	1	...	...	...	41
·07 ·08	...	...	...	...	...	3	2	3	5	5	4	2	...	1	1	...	26
·09 ·10	...	...	...	...	...	...	...	...	2	5	..	..	1	...	...	...	8
·11 ·12	...	...	...	...	1	...	..	1	1	...	..	..	...	...	...	...	3
·13 ·14	...	...	...	...	...	...	...	...	...	...	...	...	1	...	...	...	1
·15 ·16	...	...	...	...	1	...	...	...	...	1	...	...	...	1	...	...	3
	1	5	8	12	24	47	49	46	81	65	31	18	7	3	1	1	399

B. MEASUREMENT.

TABLE III.

Distribution of Parallaxes Π_0 *and* Π.

		−·20 −·21	−·19 −·18	−·17 −·16	−·15 −·14	−·13 −·12	−·11 −·10	−·09 −·08	−·07 −·06	−·05 −·04	−·03 −·02	−·01 ·00	+·01 +·02	+·03 +·04	+·05 +·06	+·07 +·08	+·09 +·10	+·11 +·12	+·13 +·14	+·15 +·16	+·17 +·18	+·19 +·20	+·21 +·22	Totals.
CALᵖ.	Π_0	1	3	3	11	31	58·5	123	226	363·5	514	607	579·5	477	280	147	86·5	43·5	20·5	8·5	1·5	...	1	3586
	Π	...	1	1	7	23	53	88	196	327	504	620	622	515	307	167	90	38	18	6	1	...	...	3584
OBSᵖ.	Π_0	·8	2·0	4·0	11·0	25·0	57·0	123·0	233·5	377·5	522·5	599·5	568·5	451·5	293	165·0	85·5	40·0	18·0	7·0	3·0	·6	·2	3587
	Π	...	·5	2·5	6·5	17·5	42·0	96·0	197·0	347·5	509·5	611·0	607·5	489·5	324·5	181·0	88·0	39·0	15·0	6·5	3·0	...	...	3584

Observations of Halley's Comet made with the Altazimuth at the Royal Observatory, Greenwich.

(*Communicated by the Astronomer Royal.*)

The following observations were obtained with the Altazimuth in azimuth 265° (*i.e.* 5° north of east). Transits over the vertical and horizontal wires were obtained both for the comet and for some neighbouring stars, from which the clock error and the azimuth error of the instrument were obtained.

G.M.T. (uncorrected for Aberration).				Apparent R.A. of Comet (corrected for Parallax).			Apparent N.P.D. of Comet (corrected for Parallax).			Assumed log Δ.
1910.	d	h	m s	h	m	s	°	′	″	
May	2	15	16 8·22	23	55	26·95	81	36	38·53	9·8643
	6	15	12 44·94	0	5	9·69	80	47	45·09	9·7557
	7	15	13 36·06	0	8	57·28	80	29	22·78	9·7408
	9	15	18 27·41	0	19	6·57	79	42	0·89	9·6739

The corrections to Zenith Distance applied for parallax are 12″·04, 15″·27, 15″·79, 18″·37 respectively, the approximate Zenith Distances being 83° 15′, 82° 12′, 81° 49′, 80° 48′.

Observer May 2, 6, Mr. Crommelin; May 7, 9, Mr. Furner.

Assumed Mean Places of Stars for 1910·0.

Star.	R.A.			N.P.D.			Days observed.
	h	m	s	°	′	″	
86 Pegasi	0	1	4·634	77	6	16·68	May 6
γ Pegasi	0	8	35·982	75	19	0·34	,, 2, 6, 7
π^1 Pegasi	22	5	14·330	57	16	3·90	,, 2
η Pegasi	22	38	46·910	60	14	59·13	,, 9
λ Pegasi	22	42	11·670	66	54	29·51	,, 9
Piazzi xxiii., 34	23	13	8·645	72	11	10·04	,, 9
W.B. xxiii., 460	23	25	59·965	73	55	57·17	,, 6, 7
71 Pegasi	23	28	57·483	67	59	53·87	,, 2
75 Pegasi	23	33	24·052	72	5	53·40	,, 2

On May 6 the nucleus of the comet appeared slightly oval, about 3″ in diameter, equivalent to 800 miles.

Royal Observatory, Greenwich:
1910 *May* 13.

Observations of Comets c 1908 *and a* 1910 *at Armagh Observatory.*
By J. L. E. Dreyer, Ph.D.

1. Comet *c* 1908 observed with a bar-micrometer on the 10-inch refractor.

1908.	G.M.T.	Δα.	Δδ.	App. R.A.	App. Decl.	Log. Par. Fact.	
	h m s	s	″	h m s	° ′ ″		
Oct. 25	7 2 32	−38·17	−300·4	19 5 2·37	+33 0 40·0	+9·2597	+0·5385
Oct. 31	7 0 19	+56·03	−328·4	18 58 34·97	+23 47 25·5	+9·3145	+0·6768
Nov. 14	6 38 46	+27·00	−198·0	18 52 0·16	+ 6 57 15·2	+9·3749	+0·8195

Each result is the mean of six measures.

Oct. 25, comet very diffused, condensation very slight. A star of the 10th mag. [B.D. +32°3332, 9·3 mag.] was at first south following the head, at $7^h\ 4^m\ 9^s$ exactly following it.

October 31, the comet has a fine tail, but very little detail is seen in the head, hardly any nucleus.

Nov. 14, I think it is brighter than I have seen it yet, tail in Pos. Angle 45°, fully 1° 40′ long. Nucleus very soft.

Mean Places of Comparison Stars for 1908·0.

	h m s	° ′ ″		Red. to App. Pl. s	Red. to App. Pl. ″
Oct. 25	19 5 40·02	+33 5 22·5	Leid. 7136	+0·52	+17·9
Oct. 31	18 57 38·41	+23 52 39·1	Berl. B. 6795	+0·53	+14·8
Nov. 14	18 51 32·58	+ 7 0 24·5	Leip. II. 8914	+0·58	+ 8·7

2. Comet *a* 1910, 10-inch refractor, powers 100 and 190.

January 21.—To the naked eye the tail is very fine, about 2° long and nearly straight, better defined on the west side ; the nucleus about 1·5 mag. In the telescope the tail is seen to be formed like a parabolic arc with a dark space between the branches and the sharp nucleus on the inner edge of the arc. Only seen for a short time.

January 24, about 5·30 p.m. Dublin time. The two branches are rather closer together than on the 21st. The nucleus is diffused, fan-shaped, with a stellar point at the apex. The coma formed by the matter issuing from the fan and turning back on both sides is distinctly broader south-following the nucleus rather than preceding it. I thought at first that the preceding branch was much the brighter, but there was probably a thin cloud over the following branch.

January 28.—The nucleus is more diffused and the nebulosity is seen all round it, also on the side away from the Sun. The preceding edge of the head seems rather more sharply defined than the following one. The fan-shape of nucleus no longer distinguishable. Tail traced very much further, probably about 20°. Only seen through breaks in clouds.

January 29.—After $6^h\ 0^m$. Best seen with power 100. The nucleus is stellar, though less sharply defined than a week ago. The fan-jet well seen, angle about 120°. I think that near the nucleus the preceding branch of the tail is the brighter. With the naked eye the tail was seen to reach α Pegasi, and even beyond it.

January 30.—Bifurcation of tail much less marked than it was a week ago, the interval between the branches being filled with nebulosity. The head is less like a sector [a sketch made at the time shows it perfectly round], and it is closer to the end of the comet nearest the Sun, very diffused. Tail about 30° long, α Pegasi quite immersed in it, and the east edge of the tail is very slightly west of ζ Pegasi.

New Double Stars. By Rev. T. E. Espin, M.A.

No.	B.D.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.		Nights.	Date. 1910.	
865	+51° 330	0 10·6	+52 2	306·0	6·65	9·3	9·5	3	·019	AB
				70·5	12·39	C=	10·5	3	·031	AC
866	+51° 58	0 17·7	+51 37	86·9	9·02	8·5	12·2	2	·001	
867	+51° 228	1 0·5	+51 45	350·5	7·00	9·2	12·2	4	·831	1909
868	+51° 386	38·2	+51 19	98·2	5·45	9·5	13·0	3	·032	
869	+51° 406	41·6	+51 23	231·6	9·32	8·6	11·6	2	·013	
870		1 41·7	+51 21	111·2	8·77	9·2	13·0	3	·032	
871		2 7·7	+55 7	299·8	3·74	9·4	10·1	3	·028	
872	+54° 551	21·6	+54 54	156·6	4·56	9·4	10·3	3	·057	AB
				18·0	7·00	C=	14·0	1	·093	AC
873	+51° 632	42·4	+51 34	98·8	4·53	9·0	11·3	4	·138	
874	+50° 697	2 58·4	+51 10	82·5	9·45	8·7	11·7	4	·025	
875	+54° 681	3 20·0	+54 59	293·0	7·30	9·4	10·5	2	·102	
876	+50° 810	37·5	+51 0	70·0	5·08	9·5	11·0	2	·083	
877	+50° 863	49·7	+50 46	73·4	6·45	8·8	10·7	3	·040	
878	+51° 817	51·3	+51 13	223·8	12·31	7·8	10·0	3	·133	
879	+50° 881	54·4	+50 47	180·1	2·58	9·3	10·7	3	·037	
880	+51° 836	3 55·3	+51 16	280·7	2·80	9·0	11·5	4	·140	
881	+54° 747	4 6·1	+55 3	250·9	7·58	9·5	9·5	3	·109	
882	+54° 774	23·4	+54 50	183·3	6·31	8·8	12·0	4	·133	
883	+54° 784	29·1	+54 16	172·8	4·20	9·4	9·6	2	·155	
884	+54° 787	30·3	+54 43	250·5	3·00	8·8	11·8	3	·127	
885	+54° 788	30·7	+54 35	157·7	6·25	8·9	12·7	3	·127	AB
				187·6	26·40	C=	10·5	3	·127	AC
886	+54° 817	39·7	+54 57	77·2	9·25	8·8	13·5	2	·166	

No.	B.D.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.	Nights.	Date. 1910.	
887	+54° 836	4 46·4	+54 57	234·6	6·05	9·5 12·5	1	·172	
888	+54° 859	4 58·2	+54 16	177·6	7·67	7·0 10·7	2	·053	AB
				246·0	26·92	C= 11·7	2	·053	AC
889	+54° 871	5 5·9	+54 17	316·3	8·20	9·3 10·7	3	·170	
890	+50° 1135	7·1	+50 47	52·3	6·05	9·6 11·7	3	·219	BC
				269·7	62·52	A= 8·5	2	·018	AB
891	+54° 892	19·2	+54 17	71·7	8 ?	9·0 12·2	2	·186	
892	+52° 963	22·5	+52 23	165·0	6·60	9·5 11·7	4	·182	
893	+51° 1098	32·0	+51 42	255·8	7·22	9·5 9·7	2	·020	
894	+52° 987	34·9	+52 33	187·0	3·22	8·9 9·2	5	·025	
895	+53° 974	50·2	+53 49	90·1	4·40	9·3 12·5	1	·172	
896		5 51·9	+53 9	331·6	2·53	10·8 11·0	4	·075	BC
				76·7	29·38	A= 9·0	4	·101	AB
				308·9	31·40	D= 9·5	3	·064	AD
897	+52° 1066	6 15·1	+52 0	85·4	6·91	8·8 11·4	3	·024	
898	+54° 1067	36·0	+54 50	302·3	8·20	8·9 12·0	3	·200	
899	+53° 1072	42·5	+53 34	300·8	3·08	9·3 10·5	3	·254	
900	+50° 1389	56·5	+50 6	51·3	10·35	8·8 9·0	3	·196	
901		57·7	+53 37	1·8	5·52	9·5 11·0	2	·219	
902	+53° 1099	6 58·1	+53 37	331·8	7·17	9·5 13·2	2	·219	
903	+50° 1428	7 16·5	+50 33	244·0	8·22	9·1 11·7	3	·166	
904	+51° 1347	36·8	+51 27	129·3	3·00	9·5 10·5	2	·192	
905	+50° 1490	49·1	+50 32	240·7	3·14	9·3 12·0	4	·179	
906		49·3	+50 34	282·0	4·77	9·8 11·0	3	·183	
907	+51° 1382	7 53·7	+51 34	109·2	6·04	9·3 10·5	3	·127	
908	+53° 1235	8 11·7	+53 47	328·1	2·95	9·4 10·7	2	·257	
909	+52° 1329	33·0	+52 6	276·3	6·40	11·5 12·7	2	·118	BC
				316·6	49·09	A= 8·0	2	·118	AB
910	+49° 1760	37·5	+49 35	12·3	9·55	9·0 13·5	2	·245	
911		8 49·2	+50 52	123·6	2·52	9·8 10·6	4	·176	
912	+48° 1763	9 17·5	+48 22	144·6	9·77	9·0 10·5	3	·239	
913	+47° 1718	47·3	+47 26	337·0	7·83	9·3 10·4	3	·240	
914	+47° 1731	9 53·6	+47 0	326·5	6·10	9·4 11·7	3	·244	
915	+48° 1857	10 14·0	+48 14	167·1	5·78	9·5 10·4	3	·210	
916		14·9	+48 54	326·8	2·97	10·2 10·2	2	·193	
917	+49° 1954	17·1	+49 13	146·5	2·26	9·0 9·3	4	·204	
918		28·7	+47 9	183·3	3·53	10·6 10·8	5	·243	
919	+46° 1645	32·3	+45 58	321·4	7·27	9·5 10·7	2	·260	
920	+47° 1827	47·8	+47 46	245·8	7·52	9·7 11·2	2	·215	BC
				125·4	61·32	A= 8·6	2	·215	AB

No.	B.D.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.	Nights.	Date. 1910.
921	+47° 1836	10 54·7	+47 15	175·5	2·70	9·5 10·0	2	·248
922	+49° 2022	11 3·1	+48 47	60·3	4·88	9·0 11·2	3	·217
923	+49° 2094	11 50·5	+48 47	219·8	2·52	9·5 10·7	3	·254
924	+48° 2037	12 30·0	+48 41	218·4	4·28	9·4 9·7	3	·254
925	+49° 4084	23 13·4	+49 19	66·5	5·97	8·8 9·7	3	·012
926		45·5	+49 50	149·5	3·30	9·2 9·7	2	·023 AB
				198·4	7·49	C= 12·0	2	·023 AC
				130·0	14·91	D= 14·0	1	·031 AD
927	+49° 4292	23 53·2	+49 29	181·8	6·97	9·3 11·8	2	·034

Notes.

881.—Argelander's place requires a correction of + 1 min. in R.A.

891.—The distance is doubtful.

Prevention of Dew-deposit upon Lens-surfaces.

By J. Franklin-Adams.

The Mervel Hill photography of the Northern Hemisphere Stars could have been completed in two years instead of nearly six had a means been known for keeping the lens-surfaces free from dew; the investigation of this question was therefore decided upon. Inquiries, letter-writing, and search in libraries gave no results.

After preliminary successful experiments with oxygen from iron bottles, an air-pump was procured from the Aerograph Co., driven by a 1 h.p. motor, and with this, air was forced through an india-rubber tube into a glass reservoir containing pumice-stone steeped in sulphuric acid, then through a tube packed with "glass wool" and thence into the air-space of a dew-cap made of two tubes; from this it escaped through holes or slits, impinging at a low velocity upon the upper surface of the upper lens.

Another tube conveys dry air into the body of the camera, whence, after drying both the lower surface of the lower lens and the gelatine side of the photographic plate, it escapes through interstices into the outer air.

The difficulties caused by dew are thus overcome.

For very delicate work, the air can be passed through a U tube or bulb containing phosphorus pentoxide, which process effectually removes the last traces of moisture.

Mervel Hill,
Hambledon, Surrey:
1910 *May* 20.

The Pressure of Radiation on Reflecting Spherical Particles.

By J. W. Nicholson, M.A., D.Sc.

(*Communicated by Professor H. F. Newall, M.A., F.R.S.*)

Since the suggestion has been revived by Arrhenius * that the force of repulsion from the Sun, which is necessary to a description of the phenomena associated with the tails of comets, might be caused by the pressure exerted upon the particles of the tails by the incident solar radiation, much discussion has taken place as to the possibility of a force of sufficient magnitude being exerted in this manner. The only mathematical analysis of the question, from a formal standpoint, has been given by Schwarzschild,† who has found that a force of the magnitude required in the case of the tails of the most difficult type (about 18·5 times the solar gravitation) could just be obtained if the size of the particles in the tails were of the same order of dimension as the mean wave-length of the incident light, and if they were perfectly reflecting. This is, of course, the case in which the pressure is a maximum, and the fact that it would need to be attained for a very large number of the particles appears to indicate that a majority of these are of the particular dimension required, and, moreover, totally reflecting. This is a very decisive result, and it is obvious that its foundations must be strictly tested. It could be avoided if the maximum were somewhat larger, or if the rise and fall of the radiation pressure towards and away from its maximum value were not very sharp, so that the range of size for an approximate maximum would be extensive. Both these possibilities are negatived by the numerical results which Schwarzschild gives, but these are few in number, and the whole question appears to demand further treatment. The analysis is very long and complicated, and in such a case the best mode of corroborating the results is to conduct another investigation with different analysis. Points of agreement in the results of the two methods may then be deemed settled. The present memoir undertakes this, proceeding by an analysis in many respects simpler and more direct, and readily adapted to numerical calculation. Schwarzschild does not give the details of his calculation after arriving at a general formula, and the results cannot therefore be extended as they stand.

The main conclusion of this paper is a support of the theory in general, and the maximum is shown to be larger. Moreover, the fall of pressure from the maximum, as the size of particle increases, is by no means so rapid as that obtained by Schwarzschild. The theory is therefore strengthened in two distinct ways.

Incidence of plane waves on a spherical obstacle. — It is convenient, at the outset, to give the spherical particle, placed in the path of the incident radiation, the most general properties, and

* *Phys. Zeit.*, ii. Jahr., Heft. vi.–vii.

† *Sitz. der Math. Phys. Classe zu Munchen*, 1901–2, p. 293.

to proceed to the special case of a total reflector at a later stage. For subsequent work, the effect of dielectric property in the obstacle will need to be examined; and as the formulæ determining the amount of diffraction and reflection can be obtained with little change in the analysis from that of the perfect reflector, this is done in the present section. The equations of the electro-magnetic theory are taken as our basis, so that a totally reflecting obstacle is the perfect conductor.

A train of plane waves incident on the sphere may be represented by Cartesian components of a magnetic vector in the form

$$a_0 = 0, \qquad b_0 = 0, \qquad c_0 = e^{ikx} \quad . \quad . \quad . \quad (1)$$

where the axis of x is the direction of propagation. There is a corresponding electric force

$$P_0 = 0, \qquad Q_0 = -Ve^{ikx}, \qquad R = 0 \quad . \quad . \quad (2)$$

where V is the velocity of propagation, and a time factor is ignored. The coordinates (x, y, z) are related to spherical polar coordinates, whose origin is the centre of the sphere by

$$x = r\cos\theta, \quad y = r\sin\theta\cos\phi, \quad z = r\sin\theta\sin\phi \quad . \quad (3)$$

This magnetic vector must be expressed in proper harmonic form. Although a general harmonic series constituting the theoretical solution has been given by Professor Love,* it will be more convenient for the present purpose to proceed otherwise, for in a problem of this nature it is better to work with the radial and transversal components of the vectors. This procedure avoids the laborious transformations inherent in Schwarzschild's method. General harmonic series for the radial and transversal components in the case of wave-motion have been developed by the writer.† Corresponding to an oscillation of wave-length $2\pi/\kappa$, it may be shown that if (a, b, c) be the components of magnetic induction along the radius vector, the tangent to the meridian, and the parallel of latitude of a sphere through the point (x, y, z) with its centre at the origin, and if (P, Q, R) be the corresponding components of electric force, then

$$\begin{aligned}
a = {} & \Sigma\Sigma n(n+1)C_n^m r^{-\frac{3}{2}}J,\ P_n^m(\mu)\cos(m\phi+\epsilon_n^m) \\
b = {} & -\Sigma\Sigma C_n^m r^{-1}\frac{d}{dr}r^{\frac{1}{2}}J,\ \sin\theta\frac{dP_n^m}{d\mu}\cdot\cos(m\phi+\epsilon_n^m) \\
& -\Sigma\Sigma mD_n^m r^{-\frac{1}{2}}J,\ \frac{P_n^m}{\sin\theta}\sin(m\phi+\delta_n^m) \\
c = {} & \Sigma\Sigma D_n^m r^{-\frac{1}{2}}J,\ \sin\theta\frac{dP_n^m}{d\mu}\cos(m\phi+\delta_n^m) \\
& -\Sigma\Sigma mC_n^m r^{-1}\frac{d}{dr}r^{\frac{1}{2}}J,\ \frac{P_n^m}{\sin\theta}\sin(m+\phi+\epsilon_n^m) \quad . \quad (4)
\end{aligned}$$

* *Proc. Lond. Math. Soc.*, xxx., 1898–9.
† *Phil. Mag.*, Feb. 1907.

Where the summations are for integral values of m and n, a factor $e^{\iota kvt}$ is omitted, and J denotes a linear combination of Bessel functions of argument kr and order $\pm(n+\frac{1}{2})$. The quantities (C, D, ϵ, δ) are arbitrary.

Moreover,

$$\iota kP/V = \Sigma\Sigma n(n+1)D_n^m r^{-\frac{3}{2}}J,\ P_n^m \cos(m\phi+\delta_n^m)$$

$$\iota kQ/V = -\Sigma\Sigma mk^2 C_n^m r^{-\frac{1}{2}}J,\ \frac{P_n^m}{\sin\theta}\sin(m\phi+\epsilon_n^m)$$
$$-\Sigma\Sigma D_n^m r^{-1}\frac{d}{dr}r^{\frac{1}{2}}J,\ \sin\theta\frac{dP_n^m}{d\mu}\cos(m\phi+\delta_n^m)$$

$$\iota kR/V = \Sigma\Sigma k^2 C_n^m r^{-\frac{3}{2}}J,\ \sin\theta\frac{dP_n^m}{d\mu}\cos(m\phi+\epsilon_n^m)$$
$$-\Sigma\Sigma mD_n^m r^{-1}\frac{d}{dr}r^{\frac{1}{2}}J,\ \frac{P_n^m}{\sin\theta}\sin(m\phi+\delta_n^m) \quad . \quad (5)$$

These are the values finite for all orientations, where $\mu=\cos\theta$. Now in the incident wave,

$$\alpha = \frac{z}{r}e^{\iota kx} = \sin\phi\sin\theta e^{\iota kr\cos\theta} \quad . \quad . \quad . \quad (6)$$

and comparing with the general possible value of α, it is seen that the only value of m for the incident wave is unity, and that for all values of n, $\epsilon_n^m=\frac{1}{2}\pi$. Also $P_n^m(\mu)=P_n^1(\mu)=\sin\theta dP_n/d\mu$ and

$$\alpha = -\sum_{n=1}^{\infty} n(n+1)C_n r^{-\frac{3}{2}}J,\ \sin\theta\sin\phi dP_n/d\mu . \quad . \quad (7)$$

where J is now $J_{n+\frac{1}{2}}(kr)$, finite at the origin.

Now, by a well-known result, due chiefly to Lord Rayleigh,

$$e^{\iota kr\cos\theta} = \Sigma_0^{\infty}(2n+1)\iota^n\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr)P_n(\mu) \quad . \quad (8)$$

and therefore on comparison, for the incident wave, after a differentiation with respect to μ

$$C_n = -\frac{2n+1}{n(n+1)}\frac{\iota^n}{\iota k}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad (9)$$

Now the electric force corresponding to (7) has only a component along the axis of y, so that D_n may be determined like C_n. For the radial component of electric force in the incident waves becomes

$$P = \frac{y}{r}Q_0 = -V\sin\theta\cos\phi e^{\iota kr\cos\theta} \quad . \quad . \quad (10)$$

or, by (8) and a differentiation with respect to $\mu=\cos\theta$,

$$P = -\frac{V}{\iota kr}\Sigma_0^{\infty}2n+1.\iota^n\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr)\cos\phi\sin\theta\frac{dP_n}{d\mu} \quad . \quad (11)$$

and comparing with (5), $m=1$ is again the only possible value, and $\delta_n^m=0$. We deduce accordingly,

$$D_n=-\iota^n.\frac{2n+1}{n.(n+1)}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}}\iota k C_n \quad . \quad . \quad . \quad (12)$$

and the radial and transversal components of electric force and magnetic induction in the incident waves are completely determined. They depend partly upon C_n and partly upon D_n, and the two portions may be treated separately. The components of magnetic induction dependent upon C_n become

$$\begin{aligned}
a&=\Sigma_1^\infty\frac{2n+1}{\iota kr}\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}\iota^n J_{n+\frac{1}{2}}(kr).\sin\theta\sin\phi\frac{dP_n}{d\mu}\\
b&=-\Sigma_1^\infty\frac{\iota^n}{\iota kr}.\frac{2n+1}{n(n+1)}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}}\frac{d}{dr}r^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr).\sin\theta\sin\phi\frac{d}{d\mu}\sin\theta\frac{dP_n}{d\mu}\\
c&=\Sigma_1^\infty\frac{\iota^n}{\iota kr}\frac{2n+1}{n(n+1)}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}}\frac{d}{dr}r^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr).\cos\phi\frac{dP_n}{d\mu} \quad . \quad . \quad (13)
\end{aligned}$$

Let the corresponding components due to the presence of the sphere be

$$\begin{aligned}
a&=-\Sigma_1^\infty n(n+1)A_n r^{-\frac{3}{2}}K_{n+\frac{1}{2}}(kr).\sin\theta\sin\phi\frac{dP_n}{d\mu}\\
b&=\Sigma_1^\infty A_n r^{-1}\frac{d}{dr}r^{\frac{1}{2}}K_{n+\frac{1}{2}}(kr).\sin\theta\sin\phi\frac{d}{d\mu}.\sin\theta\frac{dP_n}{d\mu} \quad . \quad (14)\\
c&=-\Sigma_1^\infty A_n r^{-1}\frac{d}{dr}r^{\frac{1}{2}}K_{n+\frac{1}{2}}(kr).\cos\phi\frac{dP_n}{d\mu}.
\end{aligned}$$

These are obviously the proper forms, as these components must be zero at infinity after the manner of a series of functions of type $K_{n+\frac{1}{2}}$ appropriate to divergent waves. A_n is to be determined.

Consider now the interior of the sphere. The relation between electric force and current is in general

$$Q_0=4\pi v_0/(4\pi\sigma+\iota nK).$$

Q_0 being a typical component of force and v_0 of current, σ is the specific conductivity, K the dielectric constant, and n the frequency of vibrations. Moreover

$$v_0=-\partial c_0/4\pi\mu\partial x=-\iota k c_0/4\pi\mu,$$

and

$$Q_0/c_0=-\iota k/\mu(4\pi\sigma+\iota nK).$$

Now in the external medium, $\sigma=0, n=kV, K\mu=V^{-2}, Q_0=-Vc_0$. Thus the old function $\iota k/V$ is replaced in the obstacle by $\mu(4\pi\sigma+\iota nK)$, and instead of k^2 we write

$$k_1^2=n^2K\mu-4\pi\mu\iota n\sigma \quad . \quad . \quad . \quad (15)$$

The vectors inside the sphere must be finite. The values corresponding may thus be assumed for this region in the form

$$\begin{aligned} a &= -\sum_1^\infty n(n+1)B_n r^{-\frac{3}{2}}J_{n+\frac{1}{2}}(k_1 r) \,.\, \sin\theta \sin\phi \frac{dP_n}{d\mu} \\ b &= \sum_1^\infty B_n r^{-1}\frac{d}{dr}r^{\frac{1}{2}}J_{n+\frac{1}{2}}(k_1 r) \,.\, \sin\theta \sin\phi \frac{d}{d\mu} \,.\, \sin\theta \frac{dP_n}{d\mu} \\ c &= -\sum_1^\infty B_n r^{-1}\frac{d}{dr}r^{\frac{1}{2}}J_{n+\frac{1}{2}}(k_1 r)\cos\phi \frac{dP_n}{d\mu} \end{aligned} \qquad (16)$$

with a corresponding part of the electrical vector of the form

$$\begin{aligned} (4\pi\sigma + \iota n K)\mu P &= 0 \\ (4\pi\sigma + \iota n K)\mu Q &= -\sum_1^\infty k_1^2 B_n r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(k_1 r) \,.\, \cos\phi \frac{dP_n}{d\mu} \\ (4\pi\sigma + \iota n K)\mu R &= -\sum_1^\infty k_1^2 B_n r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(k_1 r)\sin\theta \sin\phi \frac{d}{d\mu} \,.\, \sin\theta \frac{dP_n}{d\mu} \end{aligned} \qquad (17)$$

These are the vectors to which the part of the incident waves dependent on C_n gives rise. In other words, they correspond to (13).

The terms involving D_n in the vectors of the incident waves are

$$\begin{aligned} a &= 0 \\ b &= -\sum_1^\infty D_n r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(kr)\sin\phi \frac{dP_n}{d\mu} \\ c &= \sum_1^\infty D_n r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(kr)\cos\phi \sin\theta \frac{d}{d\mu} \,.\, \sin\theta \frac{dP_n}{d\mu} \\ \iota k P/V &= \sum_1^\infty n(n+1)D_n r^{-\frac{3}{2}}J_{n+\frac{1}{2}}(kr)\sin\theta \cos\phi \frac{dP_n}{d\mu} \\ \iota k Q/V &= -\sum_1^\infty D_n r^{-1}\frac{d}{dr}r^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr)\cos\phi \sin\theta \frac{d}{d\mu} \,.\, \sin\theta \frac{dP_n}{d\mu} \\ \iota k R/V &= -\sum_1^\infty D_n r^{-1}\frac{d}{dr}r^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr)\sin\phi \frac{dP_n}{d\mu} \,. \end{aligned} \qquad (18)$$

The effect of the sphere at external points may be found in general by substituting (A_n, E_n) for (C_n, D_n) respectively, and the symbol K for J, and at internal points by substituting (B_n, F_n) for (C_n, D_n) respectively, k_1 for k, and $\mu(4\pi\sigma + \iota n K)$ for $\iota k/V$.

The constants (A_n, E_n, B_n, F_n) are to be found by the surface conditions, which are conveniently satisfied by making the tangential components of both vectors continuous.

Thus the total components (b, c, Q, R) are continuous when $r = a$, if a be the radius of the sphere. Accordingly, writing briefly

$$J_{n+\frac{1}{2}}(ka) = J, \qquad K_{n+\frac{1}{2}}(ka) = K, \qquad J_{n+\frac{1}{2}}(k_1 a) = J_1 \quad . \qquad (19)$$

then, from the continuity of b, we obtain the two relations (using the *complete* vector dependent on both C_n and D_n)

$$C_n\frac{d}{da}a^{\frac{1}{2}}J + A_n\frac{d}{da}a^{\frac{1}{2}}K = B_n\frac{d}{da}a^{\frac{1}{2}}J_1$$

$$D_nJ + E_nK = F_nJ_1$$

which also secure the continuity of c. From the similar property of Q,

$$C_nJ + A_nK = \delta_2 B_nJ_1$$

$$D_n\frac{d}{da}a^{\frac{1}{2}}J + E_n\frac{d}{da}a^{\frac{1}{2}}K = \delta_1 F_n\frac{d}{da}a^{\frac{1}{2}}J_1$$

where

$$\delta_1 = \iota k^2/\mu n(4\pi\sigma + \iota nK) = k^2/k_1^2$$

$$\delta_2 = \iota k_1^2/\mu n(4\pi\sigma + \iota nK) = 1 \quad . \quad . \quad . \quad (20)$$

This secures also the continuity of R. Solving for the constants,

$$\frac{A_n}{C_n} = -\left(J\frac{d}{da}a^{\frac{1}{2}}J_1 - J_1\frac{d}{da}a^{\frac{1}{2}}J\right)\Big/\left(K\frac{d}{da}a^{\frac{1}{2}}J_1 - J_1\frac{d}{da}a^{\frac{1}{2}}K\right)$$

$$\frac{E_n}{D_n} = -\left(J_1\frac{d}{da}a^{\frac{1}{2}}J - \delta_1 J\frac{d}{da}a^{\frac{1}{2}}J_1\right)\Big/\left(J_1\frac{d}{da}a^{\frac{1}{2}}K - K\frac{d}{da}a^{\frac{1}{2}}J_1\right) \quad (21)$$

Moreover, as shown above,

$$D_n = \iota kC_n = -\iota^n\frac{2n+1}{n(n+1)}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}} . \quad . \quad . \quad (22)$$

and the disturbance radiated from the sphere is now completely determined. This general result has not hitherto been given. With these values for the constants, the total components at external points are

$$a = -\sum_1^\infty n(n+1)r^{-\frac{3}{2}}\{A_nK_{n+\frac{1}{2}}(kr) + C_nJ_{n+\frac{1}{2}}(kr)\}\sin\theta\sin\phi\frac{dP_n}{d\mu}$$

$$b = \sum_1^\infty r^{-1}\left\{A_n\frac{d}{dr}r^{\frac{1}{2}}K_{n+\frac{1}{2}}(kr) + C_n\frac{d}{dr}r^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr)\right\}\sin\theta\sin\phi\frac{d}{d\mu}.\sin\theta\frac{dP_n}{d\mu}$$
$$-\sum_1^\infty r^{-\frac{1}{2}}\{E_nK_{n+\frac{1}{2}}(kr) + D_nJ_{n+\frac{1}{2}}(kr)\}\sin\phi\frac{dP_n}{d\mu}$$

$$c = -\sum_1^\infty r^{-1}\left\{A_n\frac{d}{dr}r^{\frac{1}{2}}K_{n+\frac{1}{2}}(kr) + C_n\frac{d}{dr}r^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr)\right\}\cos\phi\frac{dP_n}{d\mu}$$
$$+\sum_1^\infty r^{-\frac{1}{2}}\{E_nK_{n+\frac{1}{2}}(kr) + D_nJ_{n+\frac{1}{2}}(kr)\}\cos\phi\sin\theta\frac{d}{d\mu}.\sin\theta\frac{dP_n}{\delta\mu}.$$

$$\iota kP/V = \sum_1^\infty n(n+1)r^{-\frac{3}{2}}\{E_nK_{n+\frac{1}{2}}(kr) + D_nJ_{n+\frac{1}{2}}(kr)\}\sin\theta\cos\phi\frac{dP_n}{d\mu}$$

$$\iota kQ/V = -\sum_1^\infty k^2r^{-\frac{1}{2}}\{A_nK_{n+\frac{1}{2}}(kr) + C_nJ_{n+\frac{1}{2}}(kr)\}\cos\phi\frac{dP_n}{d\mu}$$

$$- \sum_1^\infty r^{-1} \left\{ \mathrm{E}_n \frac{d}{dr} r^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(kr) + \mathrm{D}_n \frac{d}{dr} r^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(kr) \right\} \cos\phi \sin\theta \frac{d}{d\mu} . \sin\theta \frac{d\mathrm{P}_n}{d\mu}$$

$$\iota k \mathrm{R}/\mathrm{V} = - \sum_1^\infty k^2 r^{-\frac{1}{2}} \{ \mathrm{A}_n \mathrm{K}_{n+\frac{1}{2}}(kr) + \mathrm{C}_n \mathrm{J}_{n+\frac{1}{2}}(kr) \} \sin\theta \sin\phi \frac{d}{d\mu} . \sin\theta \frac{d\mathrm{P}_n}{d\mu}$$

$$- \sum_1^\infty r^{-1} \left\{ \mathrm{E}_n \frac{d}{dr} r^{\frac{1}{2}} \mathrm{K}_{n+\frac{1}{2}}(kr) + \mathrm{D}_n \frac{d}{dr} r^{\frac{1}{2}} \mathrm{J}_{n+\frac{1}{2}}(kr) \right\} \sin\phi \frac{d\mathrm{P}_n}{d\mu} \quad . \quad (23)$$

corresponding to the incident waves, expressed in Cartesian coordinates,

$$a_0 = 0, \quad b_0 = 0, \quad c_0 = e^{\iota kx}, \quad \mathrm{P}_0 = 0, \quad \mathrm{Q}_0 = - \mathrm{V} e^{\iota kx}, \quad \mathrm{R}_0 = 0 \quad (24)$$

When there is no conduction, $\delta_1 = \mathrm{V}_1^2/\mathrm{V}^2$ where V_1 is the velocity of propagation in the obstacle.

Case of a perfect conductor.—If the conductivity be infinite, the coefficients become more simply,

$$\mathrm{A}_n = - \mathrm{C}_n \mathrm{J}/\mathrm{K}, \qquad \mathrm{E}_n = - \mathrm{D}_n \frac{d}{da} a^{\frac{1}{2}} \mathrm{J} \Big/ \frac{d}{da} a^{\frac{1}{2}} \mathrm{K} \quad . \quad (25)$$

and the components (a, Q, R) are zero at the surface. Thus the magnetic force at the surface is tangential, and the electric force normal, as they should be. Writing as before

$$\mathrm{J} = \mathrm{J}_{n+\frac{1}{2}}(ka), \qquad \mathrm{K} = \mathrm{K}_{n+\frac{1}{2}}(ka)$$

the surface values of the other components become at once

$$\mathrm{P} = - \frac{\mathrm{V}}{\iota ka} \left(\frac{\pi}{2ka} \right)^{\frac{1}{2}} \sin\theta \cos\phi \sum_1^\infty (2n+1) \iota^n . \left\{ \mathrm{J} - \mathrm{K} \frac{d}{da} a^{\frac{1}{2}} \mathrm{J} \Big/ \frac{d}{da} a^{\frac{1}{2}} \mathrm{K} \right\} \frac{d\mathrm{P}_n}{d\mu}$$

$$b = - \frac{1}{\iota ka} \left(\frac{\pi}{2ka} \right)^{\frac{1}{2}} \sin\phi \sum_1^\infty \iota^n . \frac{2n+1}{n(n+1)} \left\{ a^{\frac{1}{2}} \frac{d}{da} a^{\frac{1}{2}} \mathrm{J} - \frac{\mathrm{J}}{\mathrm{K}} . a^{\frac{1}{2}} \frac{d}{da} a^{\frac{1}{2}} \mathrm{K} \right\} \sin\theta \frac{d}{d\mu} . \sin\theta \frac{d\mathrm{P}_n}{d\mu}$$

$$+ \left(\frac{\pi}{2ka} \right)^{\frac{1}{2}} \sin\phi \sum_1^\infty \iota^n \frac{2n+1}{n(n+1)} \left\{ \mathrm{J} - \mathrm{K} \frac{d}{da} a^{\frac{1}{2}} \mathrm{J} \Big/ \frac{d}{da} a^{\frac{1}{2}} \mathrm{K} \right\} \frac{d\mathrm{P}_n}{d\mu}$$

$$c = \frac{1}{\iota ka} \left(\frac{\pi}{2ka} \right)^{\frac{1}{2}} \cos\phi \sum_1^\infty \iota^n \frac{2n+1}{n(n+1)} \left\{ a^{\frac{1}{2}} \frac{d}{da} a^{\frac{1}{2}} \mathrm{J} - \frac{\mathrm{J}}{\mathrm{K}} a^{\frac{1}{2}} \frac{d}{da} a^{\frac{1}{2}} \mathrm{K} \right\} \frac{d\mathrm{P}_n}{d\mu}$$

$$- \left(\frac{\pi}{2ka} \right)^{\frac{1}{2}} \cos\phi \sum_1^\infty \iota^n . \frac{2n+1}{n(n+1)} \left\{ \mathrm{J} - \mathrm{K} \frac{d}{da} a^{\frac{1}{2}} \mathrm{J} \Big/ \frac{d}{da} a^{\frac{1}{2}} \mathrm{K} \right\} \sin\theta \frac{d}{d\mu} . \sin\theta \frac{d\mathrm{P}_n}{d\mu} \quad (26)$$

But, the function K being defined, when of order s, by

$$\mathrm{K}_s(ka) = \frac{\pi}{2 \sin s\pi} \{ e^{-\frac{1}{2}\iota s\pi} \mathrm{J}_{-s}(ka) - e^{\frac{1}{2}\iota s\pi} \mathrm{J}_s(ka) \} \quad . \quad (27)$$

the relation

$$a^{\frac{1}{2}} \mathrm{J} \frac{d}{da} a^{\frac{1}{2}} \mathrm{K} - a^{\frac{1}{2}} \mathrm{K} \frac{d}{da} a^{\frac{1}{2}} \mathrm{J} = \iota e^{-\iota n\pi} . \quad . \quad . \quad (28)$$

is valid, and therefore for points on the surface of the sphere,

$$P=\frac{V}{\iota ka}\left(\frac{\pi}{2ka}\right)^{\frac{1}{2}}\sin\theta\cos\phi\sum_1^\infty(2n+1)\iota^n\left(\iota a^{\frac{1}{2}}\frac{d}{da}a^{\frac{1}{2}}\mathrm{K}e^{\iota n\pi}\right)^{-1}\frac{d\mathrm{P}_n}{d\mu}$$

$$b=-\frac{1}{\iota ka}\left(\frac{\pi}{2ka}\right)^{\frac{1}{2}}\sin\phi\sum_1^\infty\iota^n\frac{2n+1}{n(n+1)}(\iota\mathrm{K}e^{\iota n\pi})^{-1}\sin\theta\frac{d}{d\mu}.\sin\theta\frac{d\mathrm{P}_n}{d\mu}$$
$$-\left(\frac{\pi}{2ka}\right)^{\frac{1}{2}}\sin\phi\sum_1^\infty\iota^n\frac{2n+1}{n(n+1)}\left(\iota a^{\frac{1}{2}}\frac{d}{da}a^{\frac{1}{2}}\mathrm{K}e^{\iota n\pi}\right)^{-1}\frac{d\mathrm{P}_n}{d\mu}$$

$$c=\frac{1}{\iota ka}\left(\frac{\pi}{2ka}\right)\cos\phi\sum_1\iota^n\frac{2n+1}{n(n+1)}(\iota\mathrm{K}e^{\iota n\pi})^{-1}\frac{d\mathrm{P}_n}{d\mu}$$
$$+\left(\frac{\pi}{2ka}\right)^{\frac{1}{2}}\cos\phi\sum_1^\infty\iota^n\frac{2n+1}{n(n+1)}\left(\iota a^{\frac{1}{2}}\frac{d}{da}a^{\frac{1}{2}}\mathrm{K}e^{\iota n\pi}\right)^{-1}\sin\theta\frac{d}{d\mu}.\sin\theta\frac{d\mathrm{P}_n}{d\mu}\quad(29)$$

Introduction of the functions R_n, ϕ_n, χ_n.—Before proceeding to a determination of the pressure, it is necessary to obtain the real portion of the vectors just found, corresponding to an incident wave of magnetic induction of the cosine type. For this purpose, we introduce the functions R_n, ϕ_n, χ_n defined by

$$\mathrm{J}_{n+\frac{1}{2}}(z)=\left(\frac{2\mathrm{R}_n}{\pi z}\right)^{\frac{1}{2}}\sin\phi_n$$
$$\mathrm{J}_{-n-\frac{1}{2}}(z)=(-)^n\left(\frac{2\mathrm{R}_n}{\pi z}\right)^{\frac{1}{2}}\cos\phi_n\qquad(30)$$
$$\tan\chi_n=-\tfrac{1}{2}\frac{\partial\mathrm{R}_n}{\partial z}$$

in the case in which $n+\frac{1}{2}$ is half an odd integer, or n an integer. The properties of these functions, and their values in various cases, may be found in a series of papers in the *Philosophical Magazine.** In the course of an investigation allied to the present,† some formulæ involving these functions were given, which may be conveniently quoted at present.

Thus, if z henceforth denotes ka,

$$\mathrm{K}_{n+\frac{1}{2}}(z)=-i(\pi\mathrm{R}_n/2z)^{\frac{1}{2}}e^{-\iota\phi_n-\iota n\pi}$$
$$\frac{d}{dz}z^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(z)=-(\pi/2\mathrm{R}_n)^{\frac{1}{2}}\sec\chi_n e^{-\iota\chi_n-\iota\phi_n-\iota n\pi}$$

where R_n, ϕ_n, χ_n are real. Introducing therefore the time factor $e^{\iota kVt}$ and taking the real part, after these substitutions, it is found that

$$P=\frac{V\sin\theta\cos\phi}{k^2a^2}\sum_1^\infty(2n+1).\mathrm{R}_n\cos(\alpha_n+\chi_n)\frac{d\mathrm{P}_n}{d\mu}$$

$$b/\sin\phi=-\frac{1}{ka}\sum_1^\infty\frac{2n+1}{n(n+1)}\left\{\frac{\sin\alpha_n}{\mathrm{R}_n^{\frac{1}{2}}}\sin\theta\frac{d}{d\mu}.\sin\theta\frac{d\mathrm{P}_n}{d\mu}-\mathrm{R}_n^{\frac{1}{2}}\cos\chi_n\sin(\alpha_n+\chi_n)\frac{d\mathrm{P}_n}{d\mu}\right\}$$

$$c/\cos\phi=\frac{1}{ka}\sum_1^\infty\frac{2n+1}{n(n+1)}\left\{\frac{\sin\alpha_n}{\mathrm{R}_n^{\frac{1}{2}}}\frac{d\mathrm{P}_n}{d\mu}-\mathrm{R}_n^{\frac{1}{2}}\cos\chi_n\sin(\alpha_n+\chi_n)\sin\theta\frac{d}{d\mu}.\sin\theta\frac{d\mathrm{P}_n}{d\mu}\right\}\quad(31)$$

* A list is given in the last paper, *Phil. Mag.*, Feb. 1910.

† "On the bending of Waves round a large Sphere," *Phil. Mag.*, April 1910. See also Macdonald, *Phil. Trans.*, A, 1910, and other papers.

where

$$\alpha_n = \phi_n + \tfrac{1}{2}n\pi + k\nabla t \quad . \quad . \quad . \quad (32)$$

and is dependent on the time.

Determination of the pressure.—As there is no electric or magnetic effect inside the conductor, the pressure exerted upon it may be deduced from the resultant of the Maxwellian tractions on its outer surface. These are—

(1) A tension along the direction of the resultant electric force.
(2) A pressure normal to this resultant.
(3) A tension along the resultant magnetic force.
(4) A pressure normal to this resultant.

Each pressure and tension is equal to the corresponding energy, electric or magnetic as the case may be, per unit volume. Since $Q = R = a = o$, these become $KP^2/8\pi$ and $\mu(b^2 + c^2)/8\pi$ respectively.

Let E be the energy per unit volume of the incident plane wave train, which will be half electric and half magnetic. Then the pressures and tensions in the present case, per unit surface, are all equal to one of the two quantities

$$EP^2/V^2 \text{ and } E(b^2 + c^2).$$

The resultant of the whole system on the unit element is a force normal to the element and equal to

$$E(b^2 + c^2 - P^2/V^2)$$

whose component along the direction of propagation of the incident wave becomes

$$E\cos\theta(b^2 + c^2 - P^2/V^2).$$

The repulsion exerted on the sphere by the radiation is therefore

$$p = \int\int E\cos\theta(b^2 + c^2 - P^2/V^2)dS$$

the integration extending over the whole surface, or

$$p = a^2E\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta\cos\theta(b^2 + c^2 - P^2/V^2)d\theta d\phi \quad . \quad (33)$$

The integration with respect to ϕ may be performed at once. Writing

$$b = b_1\sin\phi, \qquad c = c_1\cos\phi, \qquad P = VP_1\cos\phi, \quad . \quad (34)$$

where $(b_1c_1P_1)$ are independent of ϕ. Then since

$$\int_0^{2\pi}(\sin^2\phi,\ \cos^2\phi)d\phi = \pi.$$

the resultant pressure becomes

$$p = \pi a^2 \mathrm{E} \int_0^{\pi} \sin\theta \cos\theta (b_1^2 + c_1^2 - \mathrm{P}_1^2) d\theta$$
$$= \pi a^2 \mathrm{E} \int_{-1}^{1} \mu d\mu (b_1^2 + c_1^2 - \mathrm{P}_1^2) \quad . \quad . \quad . \quad (35)$$

where $\mu = \cos\theta$.

Some integral properties of zonal harmonics.—In order that the integration with respect to θ may be performed, we must consider the values of the integrals

$$\mathrm{I}_1(m, n) = \int_{-1}^{1} \mu \frac{d\mathrm{P}_m}{d\mu} \frac{d\mathrm{P}_n}{d\mu} d\mu$$
$$\mathrm{I}_2(m, n) = \int_{-1}^{1} \mu(1-\mu^2) \left\{ \frac{d}{d\mu} \sin\theta \frac{d\mathrm{P}_m}{d\mu} \right\} \left\{ \frac{d}{d\mu} \sin\theta \frac{d\mathrm{P}_n}{d\mu} \right\} d\mu \quad (36)$$
$$\mathrm{I}_3(m, n) = \int_{-1}^{1} \mu \sin\theta \frac{d\mathrm{P}_m}{d\mu} \frac{d}{d\mu}\left(\sin\theta \frac{d\mathrm{P}_n}{d\mu} \right) d\mu$$
$$\mathrm{I}_4(m, n) = \int_{-1}^{1} \mu(1-\mu^2) \frac{d\mathrm{P}_m}{d\mu} \frac{d\mathrm{P}_n}{d\mu} d\mu$$

for all integral values of m and n. For convenience, we may suppose that m is not less than n.

On integration by parts,

$$\mathrm{I}_1(m, n) = \left[\mu \mathrm{P}_m \frac{d\mathrm{P}_n}{d\mu} \right]_{-1}^{1} - \int_{-1}^{1} \left(\mathrm{P}_m \frac{d\mathrm{P}_n}{d\mu} + \mu \mathrm{P}_m \frac{d^2\mathrm{P}_n}{d\mu^2} \right) d\mu.$$

Now $d\mathrm{P}_n/d\mu$ and $\mu d^2\mathrm{P}_n/d\mu^2$ may each be expressed in series of zonal harmonics whose highest order is $n-1$. The product of each term with P_m, since $m > n-1$, will vanish on integration. Thus the integral on the right vanishes, and

$$\mathrm{I}_1(m, n) = \left[\mu \mathrm{P}_m \frac{d\mathrm{P}_n}{d\mu} \right]_{-1}^{1} = \tfrac{1}{2} n(n+1)(1 - (-)^{m+n}) \quad . \quad (37)$$

where n is the smaller of the two numbers. When $m = n$, the integral is zero.

Again,

$$\mathrm{I}_2(m, n) = \int_{-1}^{1} \mu \left\{ 1 - \mu^2 . \frac{d^2\mathrm{P}_n}{d\mu^2} - \mu \frac{d\mathrm{P}_n}{d\mu} \right\} \left\{ 1 - \mu^2 . \frac{d^2\mathrm{P}_m}{d\mu^2} - \mu \frac{d\mathrm{P}_m}{d\mu} \right\} d\mu$$
$$= \int_{-1}^{1} \mu \left\{ \mu \frac{d\mathrm{P}_n}{d\mu} - n(n+1)\mathrm{P}_n \right\} \left\{ \mu \frac{d\mathrm{P}_m}{d\mu} - m(m+1)\mathrm{P}_m \right\} d\mu$$

(by the use of the differential equation for zonal harmonics)

$$= -m(m+1) \int_{-1}^{1} \mathrm{P}_m \left(\frac{d\mathrm{P}_n}{d\mu} - n^2 \mu \mathrm{P}_n - n\mathrm{P}_{n-1} \right) d\mu + \int_{-1}^{1} \mu \frac{d\mathrm{P}_m}{d\mu} \left(\frac{d\mathrm{P}_n}{d\mu} - n^2 \mu \mathrm{P}_n - n\mathrm{P}_{n-1} \right) d\mu$$

(since $(\mu^2 - 1) . d\mathrm{P}_n/d\mu = n\mu\mathrm{P}_n - n\mathrm{P}_{n-1}$)

$$= n^2 m(m+1)\int_{-1}^{1}\mu P_m P_n d\mu + \int_{-1}^{1}\mu\frac{dP_m}{d\mu}\left(\frac{dP_n}{d\mu} - n^2\mu P_n - nP_{n-1}\right)d\mu$$

rejecting certain integrals which vanish at once since $m >= n$.

The second of these integrals, treated by parts, becomes

$$\left[P_m\left(\mu\frac{dP_n}{d\mu} - n^2\mu^2 P_n - n\mu P_{n-1}\right)\right]_{-1}^{1}$$
$$- \int_{-1}^{1} P_m\left(\frac{dP_n}{d\mu} + \mu\frac{d^2P_n}{d\mu^2} - 2\mu n^2 P_n - n^2\mu^2\frac{dP_n}{d\mu} - nP_{n-1} - n\mu\frac{dP_{n-1}}{d\mu}\right)d\mu,$$

or rejecting certain evanescent integrals as before,

$$-\tfrac{1}{2}n(n+1)(1-(-)^{m+n}) + n^2\int_{-1}^{1} P_m\left(2\mu P_n + \mu^2\frac{dP_n}{d\mu}\right)d\mu,$$

so that

$$I_2(m, n) = -\tfrac{1}{2}n(n+1)(1-(-)^{m+n})$$
$$+ n^2 m(m+1)\int_{-1}^{1}\mu P_m P_n d\mu + n^2\int_{-1}^{1}\left(2\mu P_n + \mu^2\frac{dP_n}{d\mu}\right)P_m d\mu.$$

But again writing

$$\mu^2\frac{dP_n}{d\mu} = \frac{dP_n}{d\mu} + n\mu P_n - nP_{n-1}$$

the integral last written becomes

$$\int_{-1}^{1} P_m\left(\frac{dP_n}{d\mu} + n + 2.\mu P_n - nP_{n-1}\right)d\mu = \int_{-1}^{1}\mu P_m P_n d\mu$$

again rejecting some evanescent integrals, and thus

$$I_2(m, n) = -\tfrac{1}{2}n(n+1)(1-(-)^{m+n}) + (n^2 m(m+1) + (n+2))\int_{-1}^{1}\mu P_m P_n d\mu.$$

Now μP_n can be expressed as a series of descending order commencing with a term $(n+1)P_{n+1}/2n+1$, so that as m is not less than n, the last integral is zero unless $m = n+1$, and in this case its value is $2(n-1)/(2n+1)(2n+3)$, and finally, with m not less than n,

$$I_2(m, n) = -\tfrac{1}{2}n(n+1)(1-(-)^{m+n}) \text{ with } m \neq n+1$$
$$I_2(m, n) = -\tfrac{1}{2}n(n+1)(1-(-)^{m+n}) + \frac{2n^2(n+1)(n+2)^2}{(2n+1)(2n+3)} \text{ with } m = n+1 \quad (38)$$

and we note that when $m > n$, and not equal to $n+1$,

$$I_1(m, n) + I_2(m, n) = 0 \quad . \quad . \quad . \quad (39)$$

but if $m = n+1$,

$$I_1(m, n) + I_2(m, n) = 2n^2(n+1)(n+2)^2/(2n+1)(2n+3) \quad (40)$$

These simple results are very curious. When $m=n$, the sum is zero. We pass now to the third integral, which may be expressed in the form

$$I_3(m,\ n)=\int_{-1}^{1}\mu P_m^1\frac{d}{d\mu}P_n^1\,d\mu.$$

Where $P_n^1=\sin\theta dP_n/d\mu$, we shall only require explicitly the value of $I_3(m,\ n)+I_3(n,\ m)$, which becomes

$$I_3(m,\ n)+I_3(n,\ m)=\int_{-1}^{1}\mu\frac{d}{d\mu}(P_n^1P_m^1)d\mu=-\int_{-1}^{1}P_n^1P_m^1d\mu$$

by an integration by parts. The latter integral is zero unless $m=n$, in which case its value is $m(m+1)2/2m+1$.

Finally,

$$I_3(m,\ n)+I_3(n,\ m)=0 \quad \text{when } m=n \qquad . \quad . \quad . \quad (41)$$

$$I_3(m,\ n)+I_3(n,\ m)=-2n(n+1)/2n+1 \quad \text{when } m=n\ . \quad (42)$$

The final integral required is

$$I_4(m,\ n)=\int_{-1}^{1}\mu(1-\mu^2)\frac{dP_m}{d\mu}\frac{dP_n}{d\mu}d\mu$$

$$=\int_{-1}^{1}(1-\mu^2)\frac{dP_n}{d\mu}\frac{dP_{m-1}}{d\mu}d\mu+m\int_{-1}^{1}(1-\mu^2)P_m\frac{dP_n}{d\mu}d\mu$$

by a recurrence formula. The first integral has been determined already, and is zero unless $m=n+1$, in which case it becomes $2n(n+1)/2n+1$. By another recurrence formula, the second integral is

$$n\int_{-1}^{1}P_mP_{n-1}d\mu-n\int_{-1}^{1}\mu P_mP_nd\mu.$$

The first of these vanishes, and the second exists only if $m=n+1$ (being greater than or equal to n), in which case its value is as above. Finally, it is found at once that, m being the greater of the two integers,

$$I_4(m,n)=0 \qquad m=n+1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$I_4(m,n)=2n(n+1)(n+2)/(2n+1)(2n+3) \qquad m=n+1 \quad (44)$$

and the integral is zero when $m=n$.

The electric series.—We proceed to find the contribution of the electric force P to the radiation pressure given by

$$p_1=-\pi a^2E\int_{-1}^{1}\mu d\mu P_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Now

$$P_1=\frac{P}{V\cos\phi}=\frac{\sin\theta}{k^2a^2}\Sigma_1^{\infty}(2n+1)R_n^{\frac{1}{2}}\cos\chi_n\cos(a_n+\chi_n)\frac{dP_n}{d\mu},$$

and therefore

$$-\frac{p_1}{\pi a^2 \mathrm{E}} = \frac{1}{k^4 a^4}\sum_1^\infty (2n+1)^2 \mathrm{R}_n \cos^2\chi_n \cos^2(\alpha_n+\chi_n)\mathrm{I}_4(n,n)$$

$$+\frac{2}{k^4 a^4}\sum_1^\infty\sum_2^\infty (2n+1)(2m+1)(\mathrm{R}_n\mathrm{R}_m)^{\frac{1}{2}} \cos\chi_n \cos\chi_m \cos(\alpha_n+\chi_n)\cos(\alpha_m+\chi_m)\mathrm{I}_4(m,$$

the second summation excluding equal values of m and n.

But $\mathrm{I}_4(n,n)=0$, and $\mathrm{I}_4(m,n)$ only exists when m and n differ by unity, and with the value deduced for it in the previous section

$$-\frac{p_1}{\pi a^2 \mathrm{E}} = \frac{4}{k^4 a^4}\sum_1^\infty n(n+1)(n+2)(\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}} \cos\chi_n \cos\chi_{n+1}\cos(\alpha_n+\chi_n)\cos(\alpha_{n+1}+\chi_{n+}$$

Now

$$2\cos(\alpha_n+\chi_n)\cos(\alpha_{n+1}+\chi_{n+1}) = \cos(\alpha_n-\alpha_{n+1}+\chi_n-\chi_{n+1})+\cos(\alpha_n+\alpha_{n+1}+\chi_n+\chi_{n+}$$

of which, since α_n, α_{n+1} both contain *kvt*, the second term is periodic while the first is not. In determining the motive pressure we therefore neglect the second, and since $\alpha_n-\alpha_{n+1}=\phi_n-\phi_{n+1}-\frac{1}{2}\pi$, we derive the formula

$$\frac{p_1}{\pi a^2 \mathrm{E}} = \frac{2}{k^4 a^4}\sum_1^\infty n(n+1)(n+2)(\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}} \cos\chi_n\cos\chi_{n+1}\sin(\phi_{n+1}-\phi_n+\chi_{n+1}-\chi_n) \quad (4$$

This will henceforth be called the *electric series* S_1.

The magnetic series.—In calculating the magnetic contribution to the radiation pressure it is useful to write, for the present,

$$\frac{2n+1}{kan(n+1)}\frac{\sin\alpha_n}{\mathrm{R}_n^{\frac{1}{2}}} = \beta_n, \qquad \frac{2n+1}{kan(n+1)}\mathrm{R}_n^{\frac{1}{2}}\cos\chi_n\sin(\alpha_n+\chi_n)=\delta_n \quad (47)$$

so that by (31)

$$b_1 = \frac{b}{\sin\phi} = -\sum_1^\infty\left(\beta_n\sin\theta\frac{d}{d\mu}.\sin\theta\frac{d\mathrm{P}_n}{d\mu}-\delta_n\frac{d\mathrm{P}_n}{d\mu}\right) \quad . \quad (48)$$

$$c_1 = \frac{c}{\cos\phi} = \sum_1^\infty\left(\beta_n\frac{d\mathrm{P}_n}{d\mu}-\delta_n\sin\theta\frac{d}{d\mu}.\sin\theta\frac{d\mathrm{P}_n}{d\mu}\right)$$

and the corresponding pressure is given by

$$p_2+p_3 = \pi a^2\mathrm{E}\int_{-1}^{1}\mu d\mu(b_1^2+c_1^2). \quad . \quad . \quad (49)$$

where p_2 and p_3 are the contributions of b and c respectively. Now

$$b_1^2 = \sum^{\infty 1}\left\{\beta_n^2\sin^2\theta\left(\frac{d}{d\mu}\sin\theta\frac{d\mathrm{P}_n}{d\mu}\right)^2+\delta_n^2\left(\frac{d\mathrm{P}_n}{d\mu}\right)^2-2\beta_n\delta_n\sin\theta\frac{d\mathrm{P}_n}{d\mu}.\frac{d}{d\mu}\sin\theta\frac{d\mathrm{P}_n}{d\mu}\right\}$$

$$+2\sum_1^\infty\sum_2^\infty\left\{\beta_n\beta_m\sin^2\theta.\frac{d}{d\mu}\sin\theta\frac{d\mathrm{P}_n}{d\mu}.\frac{d}{d\mu}\sin\theta\frac{d\mathrm{P}_m}{d\mu}+\delta_n\delta_m\frac{d\mathrm{P}_n}{d\mu}\frac{d\mathrm{P}_m}{d\mu}\right\}$$

$$-2\sum_1^\infty\sum_2^\infty\left\{\beta_n\delta_m\frac{d\mathrm{P}_m}{d\mu}.\sin\theta\frac{d}{d\mu}\sin\theta\frac{d\mathrm{P}_n}{d\mu}+\beta_m\delta_n\frac{d\mathrm{P}_n}{d\mu}.\sin\theta\frac{d}{d\mu}\sin\theta\frac{d\mathrm{P}_m}{d\mu}\right\}$$

the double summations being taken for unequal values of m and n only. Thus with the notation of the section dealing with zonal harmonic integrals, the contribution of b^2 to the pressure is given by

$$\frac{p_2}{\pi a^2 E} = \Sigma_1^\infty \{\beta_n^2 I_2(n, n) + \delta_n^2 I_1(n, n) - 2\beta_n \delta_n I_3(n, n)\}$$

$$+ 2\Sigma_1^\infty \Sigma_2^\infty \{\beta_n \beta_m I_2(m, n) + \delta_n \delta_m I_1(m, n) - \beta_n \delta_m I_3(m, n) - \beta_m \delta_n I_3(n, m)\}$$

remembering that $I_3(m, n)$ and $I_3(n, m)$ are not identical after the manner of the other three types of integral. Moreover, $I_1(n, n)$ and $I_2(n, n)$ are zero.

The corresponding contribution of c^2 to the pressure may be derived by an interchange of β_n with δ_n, and of β_m with δ_m, by inspection of (48) and becomes accordingly

$$\frac{p_3}{\pi a^2 E} = -2\Sigma_1^\infty \beta_n \delta_n I_3(n, n)$$

$$+ 2\Sigma_1^\infty \Sigma_2^\infty \{\delta_n \delta_m I_2(m, n) + \beta_n \beta_m I_1(m, n) - \beta_m \delta_n I_3(m, n) - \beta_n \delta_m I_3(n, m)\},$$

so that on addition

$$\frac{p_2 + p_3}{\pi a^2 E} = -4\Sigma_1^\infty \beta_n \delta_n I_3(n, n) + 2\Sigma_1^\infty \Sigma_2^\infty (\beta_n \beta_m + \delta_n \delta_m)(I_1(m, n) + I_2(m, n))$$

$$- 2\Sigma_1^\infty \Sigma_2^\infty \beta_n \delta_m + \beta_m \delta_n)(I_3(m, n) + I_3(n, m)) \quad . \quad (50)$$

with the previous restriction on the double summations, we may suppose that, in any term of these summations, m denotes the greater integer, and thus, by a property of the sum of the integrals I_1 and I_2 embodied in (39), all the terms of the first double summation vanish except those for which $m = n + 1$.

Again, when $m \neq n$, it has been shown in (41) that

$$I_3(m, n) + I_3(n, m) = 0,$$

so that the second double summation vanishes identically.

Further, m being $n + 1$, we have seen in (40) that

$$I_1(m, n) + I_2(m, n) = 2n^2(n + 1)(n + 2)^2/(2n + 1)(2n + 3),$$

and since also by (42)

$$I_3(n, n) = -n(n + 1)/2n + 1,$$

we finally obtain

$$\frac{p_2 + p_3}{\pi a^2 E} = 4\Sigma_1^\infty \frac{n(n + 1)}{2n + 1} \beta_n \delta_n + 4\Sigma_1^\infty \frac{n^2(n + 1)(n + 2)^2}{(2n + 1)(2n + 3)} (\beta_n \beta_{n+1} + \delta_n \delta_{n+1}) \quad (51)$$

a remarkably simple result.

We now recall the values of β_n and δ_n in (47), and on substitution of these,

$$\beta_n\delta_n = \frac{(2n+1)^2}{2k^2a^2n^2(n+1)^2}\cos\chi_n\{\cos\chi_n - \cos(2\alpha_n+\chi_n)\},$$

and as α_n involves the time, the second term is periodic, and may be ignored. Thus the first summation of $(p_2+p_3)/\pi a^2\mathrm{E}$ is a series

$$\mathrm{S}_2 = \frac{2}{k^2a^2}\sum_1^\infty \frac{2n+1}{n(n+1)}\cos^2\chi_n \quad . \quad . \quad . \quad (52)$$

This will be called the *first magnetic series.*
Again,

$$\beta_n\beta_{n+1} = \frac{(2n+1)(2n+3)}{k^2a^2n(n+1)^2(n+2)}\frac{\sin\alpha_n\sin\alpha_{n+1}}{(\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}}} = \frac{(2n+1)(2n+3)}{2k^2a^2n(n+1)^2(n+2)}\cdot\frac{\cos(\alpha_{n+1}-\alpha_n)}{(\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}}}$$

$$\delta_n\delta_{n+1} = \frac{(2n+1)(2n+3)}{2k^2a^2n(n+1)^2(n+2)}\cdot(\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}}\cos\chi_n\cos\chi_{n+1}\cos(\alpha_{n+1}-\alpha_n+\chi_{n+1}-\chi_n),$$

rejecting the periodic term in each case. Since also

$$\alpha_{n+1}-\alpha_n = \phi_{n+1}-\phi_n+\tfrac{1}{2}\pi,$$

we have

$$\beta_n\beta_{n+1}+\delta_n\delta_{n+1} = \frac{-(2n+1)(2n+3)}{2k^2a^2n(n+1)^2(n+2)}\left\{\frac{\sin(\phi_{n+1}-\phi_n)}{(\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}}}\right.$$
$$\left. +(\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}}\cos\chi_n\cos\chi_{n+1}\sin(\phi_{n+1}-\phi_n+\chi_{n+1}-\chi_n)\right\}$$

and the second summation of $(p_2+p_3)/\pi a^2\mathrm{E}$ is a series S_3 where

$$\mathrm{S}_3 = \frac{-2}{k^2a^2}\sum_1^\infty\frac{n(n+2)}{n+1}\left\{\frac{\sin(\phi_{n+1}-\phi_n)}{\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}}}\right.$$
$$\left. +(\mathrm{R}_n\mathrm{R}_{n+1})^{\frac{1}{2}}\cos\chi_n\cos\chi_{n+1}\sin(\phi_{n+1}-\phi_n+\chi_{n+1}-\chi_n)\right\} \quad (53)$$

This will be called the *second magnetic series.*

With the values of S_1, S_2, S_3 in (46, 52, 53), the total pressure is

$$p = \pi a^2\mathrm{E}(\mathrm{S}_1+\mathrm{S}_2+\mathrm{S}_3).$$

Further reduction of the series.—The series S_1 and S_3 may be reduced to simpler forms by means of certain properties of the functions which have not yet been given. In the first place it is desirable to eliminate the functions ϕ from the series, and this may be done by means of a remarkable relation proved below.

From the recurrence formulæ for the Bessel functions of order $n+\frac{1}{2}$ (written m for brevity), of the forms

$$\frac{2m}{z}\mathrm{J}_m = \mathrm{J}_{m-1}+\mathrm{J}_{m+1} \qquad \frac{-2m}{z}\mathrm{J}_{-m} = \mathrm{J}_{-m-1}+\mathrm{J}_{-m+1} \quad . \quad (54)$$

we deduce at once, with the asymptotic substitution (30), the relations

$$\frac{2m}{z}R_n^{\frac{1}{2}}\sin\phi_n = R_{n-1}^{\frac{1}{2}}\sin\phi_{n-1} + R_{n+1}^{\frac{1}{2}}\sin\phi_{n+1}$$

$$\frac{2m}{z}R_n^{\frac{1}{2}}\cos\phi_n = R_{n-1}^{\frac{1}{2}}\cos\phi_{n-1} + R_{n+1}^{\frac{1}{2}}\cos\phi_{n+1} \quad . \quad (55)$$

from which it may be shown that

$$(R_{n-1}R_n)^{\frac{1}{2}}\sin(\phi_n - \phi_{n-1}) = (R_nR_{n+1})^{\frac{1}{2}}\sin(\phi_{n+1} - \phi_n).$$

Similarly, by an inductive process, each is equal to

$$(R_1R_0)^{\frac{1}{2}}\sin(\phi_1 - \phi_0)$$

or -1. Thus we obtain the elegant formula

$$\sin(\phi_{n+1} - \phi_n) = -(R_nR_{n+1})^{-\frac{1}{2}} \quad . \quad . \quad (56)$$

and in the form most suitable for tabulation, the series finally become

$$S_1 = \frac{2}{k^4a^4}\sum_1^\infty n(n+1)(n+2)\left[\frac{(T_{n+1} - T_n)(R_nR_{n+1} - 1)^{\frac{1}{2}} - (1 + T_nT_{n+1})}{(1 + T_n^2)(1 + T_{n+1}^2)}\right.$$

$$S_2 = \frac{2}{k^2a^2}\sum_1^\infty \frac{2n+1}{n(n+1)}\frac{1}{1+T_n^2}$$

$$S_3 = \frac{2}{k^2a^2}\sum_1^\infty \frac{n(n+2)}{n+1}\left[\frac{1}{R_nR_{n+1}} + \frac{1 + T_nT_{n+1} - (T_{n+1} - T_n)(R_nR_{n+1} - 1)^{\frac{1}{2}}}{(1 + T_n^2)(1 + T_{n+1}^2)}\right] \quad (57)$$

where $T_n = \tan\chi_n$, $T_{n+1} = \tan\chi_{n+1}$, and the pressure is

$$p = \pi a^2 E(S_1 + S_2 + S_3) \quad . \quad . \quad . \quad (58)$$

Recurrence formulæ for R_n and χ_n.—The pressure is now expressed in terms of functions of types R_n and χ_n only. These possess interesting recurrence formulæ, which give great assistance in the numerical work. The series are found to converge fairly rapidly even near $ka = z = 1$.

Write $n = \frac{1}{2} = m$ for the present. Then from the relations (55),

$$R_{n+1}^{\frac{1}{2}}\sin\phi_{n+1} = \frac{2m}{z}R_n^{\frac{1}{2}}\sin\phi_n - R_{n-1}^{\frac{1}{2}}\sin\phi_{n-1}$$

$$R_{n+1}^{\frac{1}{2}}\cos\phi_{n+1} = \frac{2m}{z}R_n^{\frac{1}{2}}\cos\phi_n - R_{n-1}^{\frac{1}{2}}\cos\phi_{n-1},$$

and therefore, squaring and adding

$$R_{n+1} = \frac{4m^2}{z^2}R_n + R_{n-1} - \frac{4m}{z}(R_nR_{n-1})^{\frac{1}{2}}\cos(\phi_n - \phi_{n-1}).$$

But by a result of the last section

$$\cos(\phi_n - \phi_{n-1}) = \left(1 - \frac{1}{R_n R_{n-1}}\right)^{\frac{1}{2}}$$

and therefore

$$R_{n+1} = R_{n-1} + \frac{4m^2}{z^2} R_n - \frac{4m}{z}(R_n R_{n-1} - 1)^{\frac{1}{2}} \quad . \quad . \quad (59)$$

giving the recurrence formula satisfied by the functions R. Moreover, since the functions are all rational of necessity, it follows that

$$R_n R_{n-1} - 1 \quad . \quad . \quad . \quad . \quad (60)$$

is always a perfect square. This supplies a very useful check on the accuracy of the subsequent numerical work, and has been continuously used in the latter.

Again,

$$2\frac{dJ_m}{dz} = J_{m-1} - J_{m+1}, \quad \frac{2m}{z} J_m = J_{m-1} - J_{m+1},$$

and therefore

$$\frac{4m}{z} J_m \frac{dJ_m}{dz} = J^2_{m-1} - J^2_{m+1}.$$

Similarly

$$\frac{4m}{z} J_{-m} \frac{dJ_{-m}}{dz} = J^2_{-m+1} - J^2_{-m-1},$$

and therefore

$$\frac{2m}{z} \frac{d}{dz} \frac{2R_n}{\pi z} = \frac{2R_{n-1}}{\pi z} - \frac{2R_{n+1}}{\pi z},$$

or since $\tan \chi_n = -\frac{1}{2} dR_n/dz$, we obtain

$$-\tan \chi_n = \frac{z}{4m}(R_{n-1} - R_{n+1}) + \frac{R_n}{2z} \quad . \quad . \quad (61)$$

and therefore by the recurrence formula already proved

$$\tan \chi_n = \frac{R_n}{z}(m - \tfrac{1}{2}) - (R_n R_{n-1} - 1)^{\frac{1}{2}},$$

and finally, in terms of n,

$$R_{n+1} = R_{n-1} + \frac{(2n+1)^2}{z^2} R_n - \frac{2(2n+1)}{z}(R_n R_{n-1} - 1)^{\frac{1}{2}}$$

$$\tan \chi_n = \frac{n}{z} R_n - (R_n R_{n-1} - 1)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Other interesting relations may be deduced, but these are sufficient for our purpose.

Final calculation.—The successive stages of the calculation

are exhibited in the tables below. These tables include certain of the intermediate quantities, which are useful for other purposes.

The first set of tables gives the values of R_n, $\tan \chi_n$ or T_n, and $(R_n R_{n-1} - 1)^{\frac{1}{2}}$ for various arguments ranging from $z = \cdot 5$ to $z = 1$, the corresponding diameter of the obstacle thus ranging from a "small" value to that for Schwarzschild's maximum pressure.

TABLE I.

Function.	$z=\cdot 5$.	$z=\cdot 6$.	$z=\cdot 7$	$z=\cdot 8$.	$z=\cdot 9$.	$z=1\cdot 0$.
R_0	1	1	1	1	1	1
R_1	5	$34/9$	$149/49$	$41/16$	$181/81$	2
R_2	157	$78\cdot\dot{7}$	44·60683	27·66016	18·421125	13
R_3	15145	5187·42	2280·81	978·545	500·37	277
R_4	3.10^6	7.10^5	$2.2\ 10^5$	$7.5\ 10^4$	30269	12746
$\tan \chi_0$	0	0	0	0	0	0
$\tan \chi_1$	8	$125/27$	$1000/343$	$125/64$	$1000/729$	1
$\tan \chi_2$	600	253·982	115·845	60·79101	34·5984	21
$\tan \chi_3$	89328	25298	9455·9	3505·03	1571·9	771
$\tan \chi_4$	24.10^6	4.10^6	12.10^5	$36.7\ 10^4$	131.10^3	49105
$(R_0R_1 - 1)^{\frac{1}{2}}$	2	$5/3$	$10/7$	$5/4$	$10/9$	1
$(R_1R_2 - 1)^{\frac{1}{2}}$	28	$17\cdot\dot{2}$	11·603498	8·359375	6·337448	5
$(R_2R_3 - 1)^{\frac{1}{2}}$	1542	$639\cdot25\dot{5}$	318·890	164·5166	96·0021	60
$(R_3R_4 - 1)^{\frac{1}{2}}$	$2\cdot1.10^5$	6.10^4	$2\cdot3.10^4$	$8\cdot5.10^3$	3890	1879

In each case the values above are given as fractions when sufficiently simple. Many, and in particular all for $z = 1$, are exact.

Let

$$\sigma_n = \frac{(T_{n+1} - T_n)(R_n R_{n+1} - 1)^{\frac{1}{2}} - (1 + T_n T_{n+1})}{1 + T_n^2 . 1 + T_{n+1}^2} \quad . \quad . \quad (63)$$

Then the values of σ_n and $(R_n R_{n+1})^{-1}$ necessary for S_1 and S_3 are given in Table II.

TABLE II.

Function.	$z=\cdot 5$	$z=\cdot 6$.	$z=\cdot 7$.	$z=\cdot 8$.	$z=\cdot 9$.	$z=1\cdot 0$.
σ_1	$10^{-3}\cdot 50323$	·00215484	·0076208	·02090864	·04695656	·088235294
σ_2	$10^{-7}\cdot 26$	$10^{-6}\cdot 50$	$10^{-5}\cdot 164$	$10^{-5}\cdot 7834$	$10^{-4}\cdot 32000$	·000109643
σ_3	$10^{-12}\cdot 1$	$10^{-10}\cdot 14$	$10^{-9}\cdot 23$	$10^{-8}\cdot 11$	$10^{-8}\cdot 70$	$10^{-7}\cdot 370$
$(R_1R_2)^{-1}$	·00127389	·00336013	·00737239	·01410852	·02429351	$1/26$
$(R_2R_3)^{-1}$	$10^{-6}\cdot 42$	$10^{-5}\cdot 245$	$10^{-5}\cdot 983$	$10^{-4}\cdot 3694$	$10^{-3}\cdot 1085$	$10^{-3}\cdot 27770$
$(R_3R_4)^{-1}$	$10^{-10}\cdot 2$	$10^{-9}\cdot 2$	$10^{-8}\cdot 2$	$10^{-7}\cdot 13$	$10^{-7}\cdot 66$	$10^{-6}\cdot 283$

Let

$$\eta = \frac{2}{k^2a^2}\sum_1^\infty \frac{n(n+2)}{n+1}\,\frac{1}{R_nR_{n+1}} \quad . \quad . \quad . \quad . \quad (64)$$

$$\zeta = \frac{2}{k^2a^2}\sum_1^\infty \frac{n(n+2)}{n+1}\left\{\frac{(n+1)^2}{k^2a^2} - 1\right\}\sigma_n \quad . \quad . \quad (65)$$

Then $\pi a^2 E(\eta - S) = S_1 + S_3$, and therefore if p be the total pressure, and S_2 the first magnetic series,

$$p/\pi a^2 E = S_2 + \eta + \zeta \quad . \quad . \quad . \quad (66)$$

The values of these functions are exhibited in the next table, the last figure being in each case correct:—

TABLE III.

Function.	$z=·5$.	$z=·6$.	$z=·7$.	$z=·8$.	$z=·9$.	$z=1·0$.
S_2	·18463	·37155	·64473	·97427	1·2870	1·50378
η	·015296	·028037	·045238	·066442	·090691	·116868
ζ	·090601	·18174	·33453	·51540	·68705	·798800
$p/\pi a^2 E$	·29055	·5813	1·0245	1·5561	2·0648	2·41945

An examination of the pressure shows that interpolation methods may be used with fair accuracy for intermediate cases, as the successive differences exhibit a degree of regularity.

The results within this range given by Schwarzschild are for the cases $z=·5$, $z=2^{-\frac{1}{2}}=·707$, and $z=1$. The value of $p/\pi a^2 E$ given in the last case is 2·42, which agrees completely with the value above. The value for $z=·707$ is 1·07. By interpolation between the cases ·7 and ·8 in the above tables, this value is obtained, and there is again complete agreement.

But the values for $z=·5$ cannot be reconciled, as Schwarzschild gives ·35 in place of ·29055. The value given by the simple formula

$$p = \frac{14}{3}\pi a^2 E z^4 \quad . \quad . \quad . \quad . \quad (67)$$

is also ·29 in this case, and the error of this formula is then only about $\frac{1}{3}$ per cent.

The case $z=1$ is very important, being taken by Schwarzschild as practically the case of maximum pressure, his maximum being 2·5. But a direct calculation, as given below, shows that the actual maximum is decidedly larger, although occurring very close to $z=1$. Schwarzschild does not appear to have made a *direct* calculation just beyond $z=1$.

Calculation beyond $z=1$.—After the case $z=1$, corresponding to a sphere whose diameter is about a third of the wave-length, the calculation soon becomes laborious on account of the number of terms which must be accounted. In order to further define

the maximum, the cases $z = 1{\cdot}1$, $1{\cdot}2$, and $1{\cdot}5$ are tabulated below. It is seen that $z = 1{\cdot}2$ is beyond the position of the maximum. The last figure is correct in all cases.

TABLE IV.

Function.	$z=1{\cdot}1$.	$z=1{\cdot}2$.	$z=1{\cdot}5$.	Function.	$z=1{\cdot}1$.	$z=1{\cdot}2$.	$z=1{\cdot}5$.
R_1	$221/121$	$61/36$	$13/9$	$\tan\chi_1$	$1000/1331$	$125/216$	$8/27$
R_2	9·6264599	7·4236i	$37/9$	$\tan\chi_2$	13·430528	8·9699075	3·259
R_3	163·7009	102·2201	32·3086	$\tan\chi_3$	406·773	228·022	53·1358
R_4	$>5.10^3$	$>10^3$	600·561	$\tan\chi_4$	$>12.10^3$	$>5.10^3$	1462·2
R_5	$>10^6$	$>5.10^5$	19981	σ_1	·142865	·205653	·365309
$(R_0R_1-1)^{\frac{1}{2}}$	$10/11$	$5/6$	$2/3$	σ_2	$3{\cdot}38\ 10^{-4}$	$9{\cdot}41\ 10^{-4}$	·0121383
$(R_1R_2-1)^{\frac{1}{2}}$	4·072125	3·402i	$20/9$	σ_3	$<10^{-5}$	$<10^{-5}$	$1{\cdot}964.10^{-5}$
$(R_2R_3-1)^{\frac{1}{2}}$	39·68450	27·52894	11·48i	ζ	1·12020	·77998	·46556
$(R_3R_4-1)^{\frac{1}{2}}$	10^3	10^3	139·29	S_2	1·5924	1·5749	1·2896
η	·16889	·17050	·24255	$p/\pi a^2E$	2·856	2·525	1·998

The values have again been given as fractions when simple, and the number of figures is in each case sufficient to ensure the accuracy of the last figure in the value of $p/\pi a^2E$.

It is thus evident that a value 2·86 may be obtained in place of Schwarzschild's 2·5. Moreover, it is probable that this is not exactly the maximum, and that the latter is about 3. An exact determination in the above manner is somewhat laborious, but the question will be treated otherwise in a later paper. Meanwhile, we may note that as Schwarzschild found a maximum ratio 18 of pressure to gravitational force from the Sun, in the tail of a comet exposed to monochromatic light, corresponding to $p/\pi a^2E = 2{\cdot}5$, it follows that the true maximum is about 6/5 of this, or nearly 22, a value which would take account of most comets yet recorded. The effect of the distribution of energy among different wave-lengths is mentioned later. We proceed to an examination of the fall of pressure as the size of particle increases further. It is found that the fall is not quite so rapid as Schwarzschild's figures would indicate, although it does very rapidly approach the value πa^2E valid for large spheres. The following short table is sufficient to indicate this.

[TABLE.

TABLE V.

Function.	$z=2$.	$z=2{\cdot}5$.	$z=3$.	Function.	$z=2$.	$z=2{\cdot}5$.	$z=3$.
R_1	1·25	1·16	1·$\dot{1}$	$\tan\chi_1$	·125	·064	·0$\dot{3}\dot{7}$
R_2	2·3125	1·7104	1·$\dot{4}$	$\tan\chi_2$	·9375	·37632	·$\dot{1}8\dot{5}$
R_3	8·828125	4·0336	2·530864	$\tan\chi_3$	8·8359375	2·41152	·90123457
R_4	79·613281	19·732544	7·61865569	$\tan\chi_4$	132·734375	22·7067904	5·882487
R_5	1382·56738	195·93735424	45·$\dot{4}$	$\tan\chi_5$	$>10^3$	329·7029	57·1605
R_6	$>10^4$	$>10^3$	482·3388	σ_1	0	−·6231069	−·86089
(R_0R_1-1)	·5	·4	·$\dot{3}$	σ_2	·171760	·390148	0
$(R_1R_2-1)^{\frac{1}{2}}$	1·375	·992	·$\dot{7}$	σ_3	·001535	·035266	·23243
$(R_2R_3-1)^{\frac{1}{3}}$	4·40625	2·4288	1·6$\dot{2}\dot{9}$	σ_4	$<10^{-5}$	·000207	·00544
$(R_3R_4-1)^{\frac{1}{4}}$	26·49219	8·86528	4·27572	S_2	·96700	·739297	·58347
$(R_4R_5-1)^{\frac{1}{5}}$	>300	62·174	18·580	ζ	·29490	·321130	·32039
η	·327437	·381091	·41616	$\frac{p}{\pi a^2 E}$	1·5893	1·4415	1·3200

The ratios of the pressure to $\pi a^2 E$ from the various sets of tables are collected below for ready reference, with the corresponding ratios of wave-length to diameter $=\pi/z$.

TABLE VI.

z.	·5.	·6.	·7.	·8.	·9.	1·0.
$\frac{\text{Wave-length}}{\text{Diameter}}$	6·28	5·24	4·49	3·93	3·49	3·14
$\frac{p}{\pi a^2 E}$	·29055	·58133	1·0245	1·5561	2·0648	2·41945

z.	1·1.	1·2.	1·5.	2.	2·5.	3.
$\frac{\text{Wave-length}}{\text{Diameter}}$	2·86	2·62	2·09	1·57	1·26	1·05
$\frac{p}{\pi a^2 E}$	2·8564	2·5254	1·9978	1·5893	1·4415	1·320

When the diameter is approximately equal to the wave-length, it is already sufficiently large for the formula $p=\pi a^2 E$ to be valid within about 30 per cent.

A complete discussion of the actual questions raised by these results in their bearing upon tails of comets of different types involves a critical examination of other magnitudes, for example the constant of solar emission, and is conveniently left for a later paper. But it will be noted that the results strongly support the theory of radiation pressure as the origin of the tails, for they lead to a somewhat higher maximum pressure than that hitherto quoted,

and to a less rapid decrease of the pressure as the size of the particle increases. When, therefore, by the use of Wien's law, the effect of the continuous solar spectrum is found, the final pressure will necessarily be larger for two distinct reasons than that given by Schwarzschild.

It does not seem possible to discover, without analysis, whether the maximum radiation pressure thus deduced for a spherical particle corresponds closely with the similar effect to be expected in the case of an obstacle of another shape. Perhaps the effect in another case might be much greater. On this point, it is hoped that some results will shortly be obtained.

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